9	a)	Find inverse Laplace transform of $\frac{2s^2-6s+5}{(s^3-6s^2+11s-6)}$.	L3	CO3	5 M
	b)	Find the inverse Laplace transform of $\frac{1}{s(s+a)^3}$.	L3	CO3	5 M
		UNIT-V			
10	a)	Obtain the Fourier series of the function $f(x) = \begin{cases} \pi x, if\ 0 \le x \le 1 \\ \pi(2-x), 1 \le x < 2 \end{cases}$	L4	CO5	5 M
	b)	Find the half-range cosine series expansion of $f(x) = x^2$ in the range $0 \le x \le \pi$.	L4	CO5	5 M
		OR			
11	a)	Using the Fourier integral representation, show that $\int_0^\infty \frac{\omega\ sinx\omega}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x} (x > 0)$.	L4	CO5	5 M
	b)	Find the Fourier transform of $e^{-a^2x^2}, a < 0$. Hence deduce that $e^{-\frac{x^2}{2}}$ is self reciprocal in respect of Fourier transform.	L4	CO5	5 M

Code: 23BS1303

II B.Tech - I Semester – Regular Examinations - DECEMBER 2024

NUMERICAL METHODS AND TRANSFORM TECHNIQUES
(MECHANICAL ENGINEERING)

Duration: 3 hours Max. Marks: 70

Note: 1. This question paper contains two Parts A and B.
2. Part-A contains 10 short answer questions. Each Question carries 2 Marks.
3. Part-B contains 5 essay questions with an internal choice from each unit. Each Question carries 10 marks.
4. All parts of Question paper must be answered in one place.

BL – Blooms Level CO – Course Outcome

PART – A

		BL	CO
1.a)	If the first two approximations x_0 and x_1 are roots of $x^3 - x - 4 = 0$ are 1 and 2, then find x_3 by bisection method.	L2	CO1
1.b)	Write the Lagrange's interpolation formula for y = f(x).	L2	CO1
1.c)	Write first and second order derivatives using Newton's forward ward difference formula at $x = x_0$.	L2	CO1
1.d)	Write boole's rule.	L2	CO1
1.e)	Write the formula for Picard's method of successive approximation.	L2	CO1
1.f)	Explain Euler's method.	L2	CO1
1.g)	Find the Laplace transform of $e^{2t}(\ cos^2t)$.	L3	CO3
1.h)	Find the inverse Laplace transform of $\frac{s+3}{s^2-4s+13}$	L3	CO3
1.i)	If $f(x) = sinx$ in $(-\pi, \pi)$, then find b_1.	L2	CO1
1.j)	Find the Fourier sine transform of $f(x) = e^{-x}$.	L2	CO1

PART – B

			BL	CO	Max. Marks
		UNIT-I			
2	a)	Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.	L3	CO2	5 M
	b)	Using Newton-Raphson method, find the real root of $x log_{10} x = 1.2$ correct to five decimal places.	L3	CO2	5 M
		OR			
3		Find y(1) and y(10) using Newton's interpolation formulae for the data.	L4	CO4	10 M

x:	3	4	5	6	7	8	9
y:	4.8	8.4	14.5	23.6	36.2	52.8	73.9

UNIT-II

			BL	CO	Max. Marks
4		Given that	L4	CO4	10 M

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$.

OR

			BL	CO	Max. Marks
5		Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i)Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 rule (iv) Weddle's rule.	L3	CO2	10 M

			BL	CO	Max. Marks
		UNIT-III			
6	a)	Using Euler's method solve for y at x = 0.1 from $\frac{dy}{dx} = x + y + xy, y(0) = 1$, taking step size h = 0.025.	L3	CO2	5 M
	b)	Using Runge-Kutta method of fourth, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with y (0) = 1 at x = 0.2.	L3	CO2	5 M
		OR			
7		Using Taylor's series method, find y at $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = xy + y^2, y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.	L3	CO2	10 M
		UNIT-IV			
8	a)	Find the Laplace transform of $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$.	L3	CO3	5 M
	b)	Using unit step function, find the Laplace transform of $f(t) = \begin{cases} sint, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$	L3	CO3	5 M
		OR			

Code: 23BS1303

PVP23

II B.Tech I Semester Regular Examinations - December 2024

Numerical Methods & Transform Techniques

(Mechanical Engineering)

Max Marks: 70

Duration: 3hrs

Scheme of Evaluation

Part-A

1) (a) Formula - 1 M
Answer - 1 M

(b) Formula - 2 M

(c) $\frac{dy}{dx}$ - 1 M
$\frac{d^2y}{dx^2}$ - 1 M

(d) Formula - 2 M

(e) Formula - 2 M

(f) Formula - 2 M

(g) $L\{\cos t\}$ - 1 M
$L\{e^{2t}\cos t\}$ - 1 M

(h) procedure - 1 M
Answer - 1 M

(i) Formula - 1 M
Answer - 1 M

(j) Formula - 1 M
Answer - 1 M

Part – B

2, (a) interval – 1 M

procedure – 3 M

Answer – 1 M

2, (b) interval – 1 M

procedure – 3 M

Answer – 1 M

3, Table – 4 M , data – 2 M

Forward Formula – 1 M , Answer – 1 M

Backward Formula – 1 M Answer – 1 M

4, Table – 4 M

$\frac{dy}{dx}$ Formula – 2 M , Answer – 1 M

$\frac{d^2y}{dx^2}$ Formula – 2 M , Answer – 1 M

5, Table – 2 M

i, Formula – 1 M, Answer – 1 M

ii, Formula – 1 M , Answer – 1 M

iii, Formula – 1 M , Answer – 1 M

iv, Formula – 1 M, Answer – 1 M

6, (a) Data – 1 M

y_1 – 1 M

y_2 – 1 M

y_3 – 1 M

y_4 – 1 M

6, (b) k_1 – 1 M

k_2 – 1 M

k_3 – 1 M

k_4 – 1 M

Answer – 1 M

(7) Data - 1 M

y_1 - 2 M

y_2 - 2 M

y_3 - 2 M

$y_4^{(P)}$ - 1 M

$y_4^{(C)}$ - 2 M

(8) (a) $L\{f(t)\}$ - 1 M

procedure - 2 M

Answer - 2 M

(b) procedure - 3 M

Answer - 2 M

(9) (a) partial fractions - 4 M

Answer - 1 M

(b) procedure - 4 M

Answer - 1 M

(10) (a) Formula - 1 M

a_0 - 1 M

a_n - 1 M

b_n = 1 M

series - 1 M

(10) (b) Formula - 1 M

a_0 - 1 M

a_n - 2 M

series - 1 M

(11) (a) Formula - 1 M

procedure - 3 M

Answer - 1 M

(11) (b) procedure - 4 M

Answer - 1 M

code: 23BS1303   PVP23

II B.Tech - I semester - Regular Examinations - December 2024

Numerical Methods AND Transform Techniques

(Mechanical Engineering)

Duration: 3 hours   Max. Marks: 70

Part-A

1) (a) If the first two approximations x_0 and x_1 are roots of $x^3-x-4=0$ are 1 and 2, then find x_3 by bisection method.

sol: Given $x_0=1$, $x_1=2$

Then $x_2 = \frac{1+2}{2} = 1.5$ — 2M

$f(1.5)<0$, $x_3 = \frac{1.5+2}{2} = 1.75$

1) b) write Lagranges interpolation formula for $y=f(x)$.

sol: If $y=f(x)$ takes the values $y_0, y_1, \ldots y_n$ corresponding to $x=x_0, x_1, x_2, \ldots, x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n \quad \text{— 2M}$$

1) c) write first and second order derivatives using Newton's forward difference formula at $x=x_0$.

sol:

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right] \quad \text{— 1M}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \quad \text{— 1M}$$

1) d) write boole's rule

sol:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\left(7y_0+32y_1+12y_2+32y_3+14y_4+32y_5+\cdots\cdots\right)$$

— 2M

1) e) write the formula for picard's method of successive approximation.

sol:
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \quad -2M$$

1) f) Explain Euler's method

sol: consider $\frac{dy}{dx} = f(x,y)$ given that $y(x_0) = y_0$

then $y_1 = y_0 + h f(x_0, y_0)$

$y_2 = y_1 + h f(x_1, y_1)$

— — —

$y_{n+1} = y_n + h f(x_n, y_n) \quad -2M$

1) g) Find the Laplace transform of $e^{2t}\cos^2 t$.

sol: $L\{\cancel{e^{2t}\cos^2 t}\}$ $L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] \quad -1M$$

$$L\{e^{2t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right] = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{s^2-4s+8}\right] -1M$$

1) h) Find the inverse Laplace transform of $\frac{s+3}{s^2-4s+13}$

sol:
$$L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+3}{(s-2)^2+9}\right\} = L^{-1}\left\{\frac{(s-2)+5}{(s-2)^2+9}\right\} \quad -1M$$

$$= e^{2t} L^{-1}\left\{\frac{s+5}{s^2+9}\right\} = e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{5}{s^2+9}\right\}\right]$$

$$= e^{2t}\left(\cos 3t + \frac{5}{3}\sin 3t\right) \quad -1M$$

1) i) If $f(x) = \sin x$ in $(-\pi, \pi)$ then find b_1.

sol:
$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \sin nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin x \sin nx\, dx \quad -1M$$

$$\cancel{= \frac{1}{2\pi}\left[\cos\right.} = \frac{1}{\pi}\int_0^{\pi} 2\sin x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\cos(1-n)x - \cos(1+n)x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\cos(1-n)\pi - \cos(1+n)\pi) - (1-1)\right]$$

$$= \frac{1}{\pi}\left[(-1)^{1-n} - (-1)^{1+n}\right] \quad -1M$$

j) Find the Fourier sine transform of $f(x) = e^{-x}$.

sol: Fourier sine transform of $f(x) = \int_0^{\infty} f(x) \sin sx \, dx$ — 1 M

$= \int_0^{\infty} e^{-x} \sin sx \, dx$

$= \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s\cos sx) \right]_0^{\infty} = \frac{s}{1+s^2}$ — 1 M

Part - B

Unit - I

2, (a) Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.

sol: Let $f(x) = x^3 - 2x - 5$

$f(2) = -1 < 0$ & $f(3) = 16 > 0$ — 1 M

$\therefore$ root lie between 2 & 3.

Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$, $f(x_1) = 16$

$x_2 = \frac{2 f(3) - 3 f(2)}{f(3) - f(2)} = \frac{2(16) - 3(-1)}{16+1} = 2.0588$

$f(x_2) = f(2.0588) = -0.3908 < 0$

root lie between 2.0588 and 3.

$x_3 = \frac{2.0588 f(3) - 3 f(2.0588)}{f(3) - f(2.0588)} = 2.0813$

$f(x_3) = f(2.0813) = -0.1468 < 0$

root lie between 2.0813 and 3

$x_4 = \frac{(2.0813) f(3) - 3 f(2.0813)}{f(3) - f(2.0813)} = 2.0862$

$f(x_4) = f(2.0862) = -0.09278 < 0$

root lie between 2.0862 and 3

$x_5 = \frac{2.0862 f(3) - 3 f(2.0862)}{f(3) - f(2.0862)} = 2.0915$

$f(2.0915) = -0.034 < 0$

root lie between 2.0915 & 3

$$x_6 = \frac{2.0915\, f(3) - 3\, f(2.0915)}{f(3) - f(2.0915)} = 2.0934$$

$f(2.0934) = -0.0128 < 0$

root lie between 2.0934 and 3

$$x_7 = \frac{2.0934\, f(3) - 3\, f(2.0934)}{f(3) - f(2.0934)} = 2.0941$$

$f(2.0941) = -0.005 < 0$

root lie between 2.0941 and 3 — 3 M

$$x_8 = \frac{2.0941\, f(3) - 3\, f(2.0941)}{f(3) - f(2.0941)} = 2.0943$$

$\boxed{\therefore \text{ root} = 2.0943}$ — 1 M

2, b, Using Newton's Raphson method find the real root of $x \log_{10}^{x} = 1.2$ correct to five decimal places.

Sol:

Let $f(x) = x \log_{10}^{x} - 1.2$

$f(1) = \cancel{0} - 1.2 < 0$

$f(2) = -0.59794 < 0$

$f(3) = 0.23136 > 0$

$\therefore$ root lie between 2 & 3. — 1 M

Let us take $x_0 = 3$

$f'(x) = \log_{10}^{x} + x \frac{1}{x} \log_{10}^{e} = \log_{10}^{x} + 0.43429$

Newton iterative method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ — 1 M

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{0.2314}{0.9114} = 2.74610$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7461 - \frac{f(2.7461)}{f'(2.7461)} = 2.7461 - \frac{0.0048}{0.87301} = 2.74060$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7406 - \frac{f(2.7406)}{f'(2.7406)} = 2.7406 + \frac{0.00004}{0.87214} = 2.74065$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.74065 - \frac{f(2.74065)}{f'(2.74065)} = 2.74065 - \frac{0.000003}{0.87214} = 2.74065$$

— 2M

$x_3 = x_4$

$\boxed{\therefore \text{root} = 2.74065}$ — 1M

3) Find $y(1)$ and $y(10)$ using Newton's interpolation formulae for the data

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Sol: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8	3.6	2.5	0.5	0
4	8.4	6.1	3	0.5	0
5	14.5	9.1	3.5	0.5	0
6	23.6	12.6	4	0.5	
7	36.2	16.6	4.5		
8	52.8	21.1			
9	73.9				

→ 4M

Here $x_0 = 3$, $x = 1$, $h = 1$ $P = -2$ — 1M

Using Newton's forward interpolation formula

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3}(0.5) = 3.1 \quad -2M$$

Using Newton's backward interpolation formula with

$x_n = 9$, $x = 10$, $h = 1$, and $P = 1$ — 1M

$$y(10) = 73.9 + \frac{1}{1} \times 21.4 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100 \quad -2M$$

Unit-II

4, Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$.

Sol: the difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.002		
		0.299		0.005			
1.5	9.75		-0.018				
		0.281					
1.6	10.031						

— 4M

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad - 2M$$

$$= \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75 \quad - 1M$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad - 2M$$

$$= \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right]$$

$$= -0.715 \quad - 1M$$

5. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

i) Trapezoidal rule   ii) Simpson's 1/3 rule

iii) Simpson's 3/8 rule   iv) Weddle's rule

sol: Let $a=0,\ b=6,\ h=1$

$$f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$y = f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027

$-2M$

i) By Trapezoidal rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right] \quad - 1M$$

$$= \frac{1}{2}\left[(1+0.027) + 2(0.5+0.2+0.1+0.05884+0.03851)\right] = 1.4108 \quad - 1M$$

ii) By Simpson's 1/3 rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right] \quad - 1M$$

$$= \frac{1}{3}\left[(1+0.027) + 4(0.5+0.1+0.0385) + 2(0.2+0.0588)\right] = 1.3662 \quad - 1M$$

iii) By Simpson's 3/8 rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right] \quad - 1M$$

$$= \frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1)\right]$$

$$= 1.3571 \quad - 1M$$

iv) By weddle's rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right] \quad - 1M$$

$$= 0.3\left[1+5(0.5)+0.2+6(0.1)+0.0588+5(0.0385)+0.027\right]$$

$$= 1.3735 \quad - 1M$$

Unit - III

6) (a) Using euler's method solve for y at $x=0.1$ from $\frac{dy}{dx} = x+y+xy$, $y(0)=1$ taking step size $h=0.025$.

Sol: Let $x_0=0$, $y_0=1$

Given $h=0.025$

$x_1 = x_0+h = 0+0.025 = 0.025$

$x_2 = x_1+h = 0.025+0.025 = 0.05$

$x_3 = x_2+h = 0.05+0.025 = 0.075$

$x_4 = x_3+h = 0.075+0.025 = 0.1 \quad - 1M$

by Euler's method $y_{n+1} = y_n + h\,f(x_n, y_n)$

$y_1 = y_0 + h\,f(x_0,y_0) = 1+(0.025)f(0,1)$

$= 1+0.025(0+1+0) = 1.025 \quad - 1M$

$y_2 = y_1 + h\,f(x_1,y_1) = 1.025+0.025\,f(0.025, 1.025)$

$= 1.025+0.025\{0.025+1.025+(0.025)(1.025)\}$

$= 1.05189 \quad - 1M$

$y_3 = y_2 + h f(x_2, y_2) = 1.05189 + 0.025 f(0.05, 1.05189)$

$= 1.05189 + 0.025 [0.05 + 1.05189 + (0.05)(1.05189)]$

$= 1.08075$ — 1M

$y_4 = y_3 + h f(x_3, y_3) = 1.08075 + 0.025 f(0.075, 1.08075)$

$= 1.08075 + 0.025 [0.075 + 1.08075 + (0.075)(1.08075)]$

$= 1.11167$ — 1M

6, (b) Using Runge-kutta method of fourth, solve

$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$ at $x = 0.2$.

<u>Sol:</u> Given $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$

Let $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$ — 1M

$y(x_1) = y_1 = ?$

$k_1 = h f(x_0, y_0) = 0.2 f(0, 1) = (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$ — 1M

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= 0.2 f(0.1, 1.1) = (0.2)\left(\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right) = 0.2\left(\frac{1.2}{1.22}\right)$

$= 0.1967$ — 1M

$k_3 = h f(x_0 + h/2, y_0 + \frac{k_2}{2}) = 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2}\right)$

$= 0.2 f(0.1, 1.09835) = 0.2\left(\frac{(1.09835)^2 - (0.1)^2}{(1.09835)^2 + (0.1)^2}\right)$

$= 0.2\left(\frac{1.19637}{1.21637}\right) = 0.1967$ — 1M

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(0 + 0.2, 1 + 0.1967) = 0.2 f(0.2, 1.1967)$

$= 0.2\left(\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right) = 0.2\left(\frac{1.3921}{1.4721}\right) = 0.1891$ — 1M

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\left(0.2 + 2(0.1967) + 2(0.1967) + 0.1891\right)$$

$$= 1.19598 \quad - 1M$$

7, Using Taylor's series method, find y at $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ continue the solution at $x = 0.4$ using Milne's method.

Sol: Given $y' = xy + y^2$,

let $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$x_1 = x_0 + h = 0.1$

$x_2 = x_1 + h = 0.2$

$x_3 = x_2 + h = 0.3$

$x_4 = x_3 + h = 0.4$

we find y_1, y_2, y_3 using Taylors series — 1 M

$y' = xy + y^2 \Rightarrow y_0' = x_0 y_0 + y_0^2 = 0 + 1^2 = 1$

$y'' = xy' + y + 2yy' \Rightarrow y_0'' = x_0 y_0' + y_0 + 2y_0 y_0'$

$= 0 + 1 + 2(1)(1) = 3$

$y''' = xy'' + y' + y' + 2yy'' + 2(y')^2$

$\Rightarrow y_0''' = x_0 y_0'' + 2y_0' + 2y_0 y_0'' + 2(y_0')^2$

$= 0 + 2(1) + 2(1)(3) + 2(1) = 10$

By Taylor's Series

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(10) + \cdots$$

$$= 1 + 0.1 + 0.015 + 0.00167 = 1.1167 \quad - 2M$$

To find y_2:

$y_1' = x_1 y_1 + y_1^2 = (0.1)(1.1167) + (1.1167)^2 = 1.3587$

$y_1'' = x_1 y_1' + y_1 + 2y_1 y_1' = (0.1)(1.3587) + 1.1167 + 2(1.1167)(1.3587)$

$= 4.2871$

$y_1''' = x_1 y_1'' + 2y_1' + 2y_1 y_1'' + 2(y_1')^2$

$= (0.1)(4.2871) + 2(1.3587) + 2(1.1167)(4.2871) + 2(1.3587)^2$

$= 0.42871 + 2.7174 + 9.5748 + 3.6921 = 16.4130$

$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 1.1167 + (0.1)(1.3587) + \frac{(0.1)^2}{2!}(4.2871) + \frac{(0.1)^3}{3!}(16.413) + \cdots$

$= 1.1167 + 0.13587 + 0.0214 + 0.0027355 + \cdots$

$y_2 = 1.2767$ — 2M

To find y_3:

~~$y_3' = x_2 y$~~ $y_2' = y_2 x_2 + y_2^2 = (1.2767)(0.2) + (1.2767)^2 = 1.8853$

$y_2'' = x_2 y_2' + y_2 + 2y_2 y_2' = (0.2)(1.8853) + 1.2767 + 2(1.2767)(1.8853)$

$= 6.4677$

$y_3 = y_2 + h y_2' + \frac{h^2}{2!} y_2'' + \cdots$

$= 1.2767 + (0.1)(1.8853) + \frac{(0.1)^2}{2}(6.4677) + \cdots$

$= 1.2767 + 0.18853 + 0.03234 = 1.4976$ — 2M

Using the predictor

$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$= 1 + \frac{4(0.1)}{3}\{2(1.1167) - \cancel{1.2767} + \cancel{2(1.4976)}\}$

$= 1.5269$ $\qquad y_3' = x_3 y_3 + y_3^2 = (0.3)(1.4976) + (1.4976)^2$

$= 2.6921$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.3587) - 1.8853 + 2(2.6921)\right]$$

$$y_4^{(P)} = 1.8288 \quad - 1M$$

$$y_4' = x_4 y_4 + y_4^2 = (0.4)(1.8288) + (1.8288)^2 = 4.07603$$

Using corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.2767 + \frac{(0.1)}{3}\left[1.8853 + 4(2.6921) + 4.07603\right]$$

$$= \cancel{1.843}\ 1.8344 \quad - 2M$$

8,(a) Find the Laplace transform of $f(t)=\begin{cases}1, & 0\le t\le 1\\ t, & 1<t\le 2\\ 0, & t>2\end{cases}$

Sol: $L\{f(t)\}=\int_0^\infty f(t)e^{-st}\,dt$ — 1 M

$$=\int_0^1 1\,e^{-st}\,dt+\int_1^2 t\,e^{-st}\,dt+\int_2^\infty 0.e^{-st}\,dt \quad — 2M$$

$$=\left[\frac{e^{-st}}{-s}\right]_0^1+\left[t\,\frac{e^{-st}}{-s}-\int\frac{e^{-st}}{-s}\,dt\right]_1^2$$

$$=\frac{e^{-s}}{-s}-\frac{1}{-s}+\left[-\frac{t\,e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_1^2$$

$$=\frac{e^{-s}}{-s}+\frac{1}{s}+\left[\frac{-2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}\right]$$

$$=\cancel{\frac{e^{-s}}{-s}}+\frac{1}{s}-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\cancel{\frac{e^{-s}}{s}}+\frac{e^{-s}}{s^2}=\frac{1}{s}-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2} \quad — 2M$$

8,(b) Using unit step function, find laplace transform of

$$f(t)=\begin{cases}\sin t, & 0\le t<\pi\\ \sin 2t, & \pi\le t<2\pi\\ \sin 3t, & t\ge 2\pi\end{cases}$$

Sol: $f(t)=\sin t\,[u(t-0)-u(t-\pi)]+\sin 2t\,[u(t-\pi)-u(t-2\pi)]+\sin 3t.u(t-2\pi)$

$$=\sin t+(\sin 2t-\sin t)\,u(t-\pi)+(\sin 3t-\sin 2t)\,u(t-2\pi) \quad — 2M$$

$$L\{f(t-a)u(t-a)\}=e^{-as}\bar{f}(s) \text{ and } L(\sin at)=\frac{a}{s^2+a^2} \quad — 1M$$

$$L\{f(t)\}=L\{\sin t\}+L\{(\sin 2t-\sin t).u(t-\pi)\}+L\{(\sin 3t-\sin 2t).u(t-2\pi)\}$$

$$=\frac{1}{s^2+1}+e^{-\pi s}\left[\frac{2}{s^2+4}-\frac{1}{s^2+1}\right]+e^{-2\pi s}\left[\frac{3}{s^2+9}-\frac{2}{s^2+4}\right] \quad — 2M$$

9) a, Find the Inverse Laplace transform of $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Sol:
$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad - 1M$$

$$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Put $s=1$

$$2-6+5 = A(-1)(-2) \Rightarrow \boxed{A = \tfrac{1}{2}}$$

Put $s=2$

$$8-12+5 = B(1)(-1) \Rightarrow \boxed{B=-1}$$

Put $s=3$

$$18-18+5 = C(2)(1) \Rightarrow \boxed{C = \tfrac{5}{2}} \quad - 3M$$

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - 1\,L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{1}{2}e^{t} - e^{2t} + \tfrac{5}{2}e^{3t} \quad - 1M$$

9, (b) Find the inverse Laplace transform of $\frac{1}{s(s+a)^3}$

Sol:
$$\cancel{L^{-1}\left\{\frac{1}{s(s+a)^3}\right\}} \quad L^{-1}\left\{\frac{1}{(s+a)^3}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^3}\right\} = e^{-at}\frac{t^2}{2!} = \frac{t^2e^{-at}}{2} \quad - 2M$$

$$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = \int_0^t \frac{t^2e^{-at}}{2}\,dt \quad - 1M$$

$$= \frac{1}{2}\left\{ t^2\frac{e^{-at}}{-a} - 2t\frac{e^{-at}}{a^2} + 2\frac{e^{-at}}{-a^3}\right\}_0^t$$

$$= \frac{1}{2}\left[\cancel{-\frac{a^2e^{-at}}{a}}\right. = \frac{1}{2}\left[-\frac{t^2e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right] \quad - 2M$$

Note: Give marks for any method

10)(a) obtain the Fourier series of the function

$$f(x) = \begin{cases} \pi x, & \text{if } 0 \le x \le 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases}$$

Sol: let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ — 1M

where $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\frac{x^2}{2}\right] + \pi\left[2x - \frac{x^2}{2}\right]_0^2$

$= \pi$ — 1M

$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$

$= \left(\pi x \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$

$= \frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi} + \left(-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$

$= 0$ if n is even

$= -\frac{4}{n^2\pi}$ if n is odd — 1M

$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$

$= \left(\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$

$= -\frac{\cos n\pi}{n} + \frac{\cos n\pi}{n} = 0$ — 1M

$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$ — 1M

10 b) Find the half range cosine series expansion of $f(x) = x^2$ in the range $0 \le x \le \pi$.

Sol:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — 1 M

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^3}{3}\right)$

$= \frac{2\pi^2}{3}$ — 1 M

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi} x^2 \cos nx\,dx\right]$

$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[0 + 2\pi\frac{(-1)^n}{n^2} - 0\right] = \frac{4(-1)^n}{n^2}$ — 2 M

$\therefore f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$

$x^2 = \frac{2\pi^2}{3} + 4\left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$ — 1 M

11 a) Using the Fourier integral representation show that

$\int_0^{\infty} \frac{\omega \sin x\omega}{1+\omega^2}\,d\omega = \frac{\pi}{2} e^{-x} \; (x > 0)$

Sol:

Fourier sine integral of $f(x)$ is — 1 M

$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin \omega x \int_0^{\infty} f(t)\sin \omega t\,dt\,d\omega$

Let $f(t) = e^{-at}$, $a > 0$ then

$e^{-x} = \frac{2}{\pi}\int_0^{\infty} \sin \omega x\left(\int_0^{\infty} e^{-t}\sin \omega t\,dt\right) d\omega$

$= \frac{2}{\pi}\left[\int_0^{\infty} \sin \omega x\left(\frac{e^{-t}}{1+\omega^2}(-\sin \omega t - \omega\cos \omega t)\right)_0^{\infty}\right] d\omega$

— 2 M

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \omega x}{1+\omega^2} d\omega$$

$$\Rightarrow \int_0^\infty \frac{\sin \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x} \quad - 2M$$

11, (b) Find the Fourier transform of $e^{-a^2x^2}, a<0$. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of Fourier transform.

Sol: $F(e^{-a^2x^2}) = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)} dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2} \cdot e^{-s^2/4a^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{-s^2/4a^2} dt/a \quad (\because a(x - is/2a^2) = t)$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \quad - 3M$$

$$F(e^{-a^2x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Taking $a^2 = 1/2$

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{1/\sqrt{2}} e^{-s^2/2} = \sqrt{2\pi}\, e^{-s^2/2} \quad - 2M$$

i.e Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$

Also the dug $e^{-x^2/2}$ and $e^{-s^2/2}$ are reciprocal.