Discrete Mathematics (UNIT-I)

What is discrete mathematics.

Definition:

Discrete mathematics is mathematics that deal with discrete objects. Discrete objects are those which are separated from (not connected to/ distinct from) each other.

i.e objects that can assume only distinct, separated values.

- The term "Discrete mathematics" is therefore used in contrast with "continuous mathematics" which is the branch of mathematics dealing with objects.

Examples of objects with discrete values

- integers, graphs/ statements in logic, automobiles, peoples, houses etc.

Discrete mathematics and Computer Science

- concepts from discrete mathematics are useful for describing objects and problems in computer algorithms and problems programming languages.
- These have applications in cryptography, automated theorem proving, and software development.

Why discrete mathematics:

Let us first see why we want to be interested the formal/ theoretical approaches in computer sci

Some of the major reasons that we adopt the formal approaches are:

1.) we can handle infinity or large quantity and indefiniteness with them.

2) results from formal approaches are reusable.

Standard applications of Discrete mathematics in Computer Science:

UNIT-I: (propositional logic).

① The design of digital circuits is entirely based on propositional logic, so much so that its engineers call it "logic design" rather than "circuit design"

② Even writing a computer program is often thought to ~~involu-~~ involve devising its "logic".

(note that "logic" in the later sense is an informal logic), used to refer to the idea rather than formal flow on information through the program and whether it is being processed correctly.

Mathematical Logic :

What is logic –

Mathematical or propositional logic. –

↓

The area of logic that deals with the ~~propo~~ proposition.

Contingency

A stat formulae which is neither a tautology nor a contradictory is called contingency.

$(p \to q) \wedge (p \wedge q)$ is a contingency.

Ex: tautology or contingency or contradictory

i) $p \to (q \to p)$ — T

ii) $(q \wedge p) \vee (q \vee \sim p)$ — C

iii) $q \vee (\sim q \wedge p)$ — C.

PT

PTL

Logic :- It deals with methods of reasoning.
The main aim of logic is to provide rules by which we can determine whether the particular reasoning / argument is valid.

✓ Greek philosopher, Aristotle was the pioneer of the logical reasoning.

Def. proposition : It is a collection of declarative stmt that has either a truth values, "true" or "false" can be assigned but not both.

✓ a, b, c, --- d —— alphabets.
✓ cat, dog, man, chair —— words.

✓ She is a girl ——
He is a boy —— } sentence.

Sentences
- Declarative — either T/F.
- Imperative — no T/F.
- Interrogative — (?)
- Exclamatory. — (!).

~~Man~~ Mostly lower case letters can be used to represent the propositions.

Mathematical Logic (UNIT - I)

Mathematical or propositional logic

The rules of mathematical logic specify methods of reasoning mathematical statements.

(or)

Logic is the discipline that deals with methods of reasoning. The main aim of Logic is to provide rules by which we can determine whether the particular reasoning or argument is valid.

→ Greek philosopher, Aristotle was the pioneer of logical reasoning.

→ logical reasoning is used to in many disciplines to establish valid results.

Rules of logic are used to provide :-

1. proofs of theorems in mathematics
2. to verify the correctness of computer programs.
3. to draw the conclusions from scientific experiments.

→ It has many practical applications in computer science like design of computing machines, artificial Intelligence, definition of data structures for programming languages etc

propositions or statements :-

Definition: (propositional logic)

It is concerned with statements to which the truth values, "true" or "false" can be assigned.

The purpose is to analyze these statements either individually or in a composite manner.

Definition:- (Proposition)

A proposition is a Collection of declarative statements that has either a truth value "true" (or) "false" can be assigned but not both.

→ Sentences which are exclamatory, interrogative or imperative in nature are not propositions.

→ Lower case letters such as p, q, r --- are used to denote the propositions.

For example; we consider the following sentences.

Valid propositions:

1. New Delhi is the capital city of india (T)
2. 2+2 = 3 (F)
3. Man is mortal. (T)
4. "1+2+9 = 3-2" (F).

invalid propositions:

1. How beautiful is rose? (interrogative)
2. Take a cup of coffee. (imperative)
3. "A is less than 2" (It is because unless we give a specific value of A, we can not say whether the statement is true or false.).

→ If the proposition is true, we say that the truth value of that proposition is true. denoted by "T" or "1".

→ If the proposition is false, the truth value is said to be false. denoted by "F" or "0".

There are two types of Propositions:

✓ ① Primitive or Primary or atomic

✓ ② compound or molecular proposition

① Primitive or Primary:

Proposition which do not contain any of the logical operator or connectives are called atomic or primitive or primary proposition.

② Compound or molecular:

combine two or mono atomic statements using connectives are called molecular or compound statement.

→ propositional logic:

The area of logic that deals with propositions is called propositional calculus or propositional logic.

Note;

The truth value of a compound statement or proposition depends on those of sub propositions.

And the way in which they are combined using connectives.

Connectives:

There are five types of connectives.

① conjunction
② disjunction
③ conditional
④ biconditional
⑤ negation

① Conjunction:

Let p and q are two statements. The statement $p \wedge q$ is called conjunction of the "p and q".

which is read as p and q.

The statement has the truth value T when both p and q have the truth value T. Otherwise it is False (F).

Truth table:-

Definition: Determine the truth value of statement formula for each possible combination of the truth values of the compound statement.

A table showing all such truth values is called the truth table of the formula.

Truth table for Conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:-

p: Canada is a country

q: moscow is the capital of spain.

$p \wedge q$: Canada is a country "and" moscow is the capital of spain.

Ex: p: Today is Monday

q: There are 50 tables in this room.

The conjunction of p and q, that is $p \wedge q$ may be written as

$p \wedge q$: Today is monday and there are 50 tables in this room.

② disjunction:

The disjunction of the statements p, q is denoted by $p \vee q$. which is read as "p or q"

The statement $p \vee q$ has ~~two~~ the truth value F only when both p and q have the truth value F. Otherwise it is T.

Truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

① $p \vee q$: Canada is a country or moscow is the capital of spain.

② $p \vee q$: Today is monday or there are 50 tables in this room.

(3) Conditional or implication;

The implication of two statements p, q is denoted by $p \to q$.

which is read as (1) if p then q

(2) p is sufficient for q.

(3) p is sufficient condition for q

(4) q is necessary for p.

(5) q is necessary condition for p.

(6) p only if q. etc.

→ The statements $p \to q$ has a truth value 'F' when p has truth value T and q has truth value F. otherwise it is T.

Truth table for implication:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The verbal translation of $p \to q$ is.

(1) $p \to q$: If canada is a country then moscow is the capital of spain.

(2) $p \to q$: If today is monday then there are 50 tables in this room.

Note: The statement 'p' is called "hypothesis" of the implication. And q is called "Conclusion".

④ Biconditional or bi-implication:

If p, q are two statements then the biconditional of two statements p, q are denoted by $p \leftrightarrow q$ or $p \rightleftarrows q$.

which is read as ① p if and only if q

② p iff q

③ p is necessary and sufficient for q.

④ If p then q, and conversely.

→ The statement $p \leftrightarrow q$ has the truth value T when both p and q have identical truth values. otherwise it has a truth value F.

Truth table for biconditional:

p	q	$p \leftrightarrow q$
T	T	T
F	F	F
F	T	F
F	F	T

Example:

① $p \leftrightarrow q$: Canada is a country if and only if moscow is a capital of spain.

② $p \leftrightarrow q$: Today is monday if and only if there are 50 tables in this room.

(5) Negation:

If p is a statement then its negation is $\neg p$ or $\sim p$.

which is read as "not p".

→ If the truth value of p is T, then truth value of $\sim p$ is F. also if ~~the~~ truth value of p is F, then the truth value of $\sim p$ is T.

Truth table for negation:

p	$\sim p$
T	F
F	T

Example:

(1) p: Canada is a country

$\sim p$: Canada is not a country.

(2) p: chennai is a city.

$\sim p$: It is not the case that Chennai is a city.

Order of precedence for logical operators:

Order of precedence for the logical operators given as follows

① The negation operator has precedence over all the logical operators. Thus

$\sim p \wedge q$ means $(\sim p)\wedge q$ but not $\sim(p\wedge q)$.

② The conjunction operator has precedence over the disjunction operator. Thus

$p\wedge q \vee r$ means $(p\wedge q)\vee r$, but not $p\wedge(q\vee r)$

③ The conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than other operators. Among them, $\rightarrow$ has precedence over $\leftrightarrow$.

Note: $(p\vee q)\wedge(\sim r)$ is the conjunction of $p\vee q$ and $\sim r$.

Construct the truth table for $(p\to q)\wedge((q\wedge \sim r)\to(p\vee r))$

p	q	r	$p\to q$ ①	$\sim r$	$q\wedge\sim r$	$p\vee r$	$q\wedge\sim r\to p\vee r$ ②	①$\wedge$②
1	1	1	1	0	0	1	1	1
1	1	0	1	1	1	1	1	1
1	0	1	0	0	0	1	1	0
1	0	0	0	1	0	1	1	0
0	1	1	1	0	0	1	1	1
0	1	0	1	1	1	0	0	0
0	0	1	1	0	0	1	1	1
0	0	0	1	1	0	0	1	1

Truth table for $\sim(p \vee (q \wedge r)) \leftrightarrow (\ldots$

Construct the truth table for $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$ ①	$\sim p$	$\sim q$	$\sim p \vee \sim q$ ②	①$\leftrightarrow$②
0	0	0	1	1	1	1	1
0	1	0	1	1	0	1	1
1	0	0	1	0	1	1	1
1	1	1	0	0	0	0	1

①$\leftrightarrow$②
0
0
1

→ Construct the truth table for the following formulas.

(i) $(q \wedge (p \to q)) \to p$

(ii) $\sim(p \vee (q \vee r)) \leftrightarrow (p \vee q) \wedge (p \vee r)$.

Truth table for $(q \wedge (p \to q)) \to p$

p	q	$p \to q$	$q \wedge (p \to q)$	$(q \wedge (p \to q)) \to p$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

1.6

Truth table for $\sim(p \vee (q \wedge r)) \leftrightarrow (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$\sim(p \vee (q \wedge r))$ ①	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$ ②	① $\leftrightarrow$ ②
0	0	0	0	0	1	0	0	0	0
0	0	1	0	0	1	0	1	0	0
0	1	0	0	0	1	1	0	0	0
0	1	1	1	1	0	1	1	1	0
1	0	0	0	1	0	1	1	1	0
1	0	1	0	1	0	1	1	1	0
1	1	0	0	1	0	1	1	1	0
1	1	1	1	1	0	1	1	1	0

Well-Formed Formulas:

4, 5, 6,

Definition:- (statement formula)

A statement formula is an expression which is a string consisting of variables, parentheses, and connective symbols. Note that every string of these symbols is a statement formula.

We shall now give a recursive definition of a statement formula, often called a well formed formula (wff).

A well formed formula can be generated by the following rules

R_1: A statement variable standing alone is a well formed formula.

R_2: If A is well formed formula, then $\sim A$ is a well formed formula.

If A and B are well formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, and $(A \leftrightarrow B)$ are well formed formulas.

Definition: (well formed formula)

A string of symbols containing the statement variables, connectives, and parenthesis is said to be well formed formula if it can be obtained by finitely many applications of rules R_1, R_2 and R_3.

Ex:- According to this definition, the following are well formed formulas.

$\sim(P \wedge Q)$, $\sim(P \vee Q)$, $(P \to (P \vee Q))$, $(P \to (Q \to R))$ and

$(((P \to Q) \wedge (Q \to R)) \rightleftarrows (P \to R))$.

Example 1:-

$\sim P \wedge Q$ is not a well formed formula because a wff would be either $(\sim P \wedge Q)$ or $\sim(P \wedge Q)$.

Here $\sim P \wedge Q$ does not contain parenthesis.

Example 2:-

$(P \to Q$ is not well formed formula.

but $(P \to Q)$ is a well formed formula (wff).

Example 3:-

$(P \wedge Q) \to Q)$ is not wff. because one of the parenthesis in the beginning is missing.

Example 4:-

$(P \to (P \vee Q))$ is a well formed formula.

Example 5:-

$((P \to (\sim P)) \to (\sim P))$ is a wff.

Example 6:

$((P \to (Q \to R))) \to ((P \to Q) \to (P \to R)))$ is not wff because one of the parenthesis in the begening is missing.

Example 7:-

$((\sim P \to Q) \to (Q \to P)))$

It is not wff because one more parenthesis in the end.

Example 8:-

$((P \wedge Q) \Leftrightarrow)$ is a wff.

- Tautologies and Contradictions

It is compound statement that is always true.

→ the truth table of a tautology will contain only 'T' entries in the last column.

Importance of tautology:

- Tautology helps conclude some statement from some given statements.

For Example:-

if the statements "$p \Rightarrow q$" is a tautology, then it is easy to conclude the truth of q from the truth of p.

Thus with the help of tautology we move from some given statement to some concluding statement in a step-by-step manner.

which is justified with in the frame work of mathematical logic.

<u>Example</u>: Show that $p \vee \sim p$ is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$p \vee \sim p$ has truth value 'T' for all its entries in the false.

$\therefore$ it is a ~~tautology~~. tautology. This is also called "Law of enclusive middle".

i.e either p is true or it is false. there is no middle possibility.

<u>Example</u>: Show that statement $(p \wedge q) \Rightarrow q$ is a tautology.

p	q	$p \wedge q$	$(p \wedge q) \rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$\therefore$ All entries in the last column has truth value

$\therefore$ it is a tautology.

Contradiction:

A Statement is said to be contradiction its truth value is always false (F) for all its entries in the truth table.

→ If the statement is a contradiction, then its negation will be a tautology.

Example: Show that $p \wedge \sim p$ is a tautology or contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	T
F	T	T

∴ Since the statement $p \wedge \sim p$ has its false value for all its entries in the truth table.

Hence it is a contradiction.

Example: Establish $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ as a tautology.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

∴ the given statement is a tautology.

Examples:- (Exercises for student)

① Show that $(p \wedge q) \Rightarrow (p \vee q)$ is tautology but $(p \vee q) \Rightarrow (p \wedge q)$ is not.

② Show that $p \vee \sim(p \wedge q)$ is a tautology.

③ Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction or tautology.

④ Make the truth table for $(p \downarrow q) \wedge (p \downarrow r)$, $(p \uparrow q \uparrow r)$ and $(p \oplus q \oplus r)$.

Equivalence of propositions (or) Logically equivalent (or) Equivalence of formulas

Two compound statements $A(p_1, p_2, \ldots p_n)$ and $B(p_1, p_2, \ldots p_n)$ are said to be logically equivalent or simply equivalent if they have identical truth tables.

If the truth value of A is ~~equival~~ equival to the truth value of B for every one of 2^n possible sets of truth values assigned to $p_1, p_2, p_3 \ldots p_n$.

The equivalence of two propositions A and B is denoted as $A \Leftrightarrow B$ or $A \equiv B$.

which is read as "'A' is equivalent to B".

Note:- $\Leftrightarrow$, $\equiv$ these are not a connectives.

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(a) Truth table method:

Ex: prove that $(p \leftrightarrow q) \Leftrightarrow (\sim p \vee q) \wedge (\sim q \vee p)$

p	q	$p \leftrightarrow q$	$\sim p$	$\sim p \vee q$ (1)	$\sim q$	$\sim q \vee p$ (2)	(1) $\wedge$ (2)
T	T	T	F	T	F	T	T
T	F	F	F	F	T	T	F
F	T	F	T	T	F	F	F
F	F	T	T	T	T	T	T

$\therefore p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

Example: Prove that $p \wedge (\sim q \vee r)$ and $p \vee (q \wedge \sim r)$ are logically equivalent or not.

Sol: $p \wedge (\sim q \vee r) \equiv p \vee (q \wedge \sim r)$.

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$ (1)	$\sim r$	$q \wedge \sim r$	$p \vee (q \wedge \sim r)$ (2)
T	T	T	F	T	T	F	F	T
T	T	F	F	F	F	T	T	T
T	F	T	T	T	T	F	F	T
T	F	F	T	F	T	T	F	T
F	T	T	F	T	F	F	F	F
F	T	F	F	F	F	T	T	T
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	T	F	F

$\therefore p \wedge (\sim q \vee r) \not\equiv p \vee (q \wedge \sim r)$.

Example: Prove that the following equivalences are.

ⓐ $\sim(p \to q) \equiv p \wedge \sim q$.

ⓑ $(p \vee q) \wedge (p \vee \sim q) \equiv p$.

ⓒ $(p \wedge q) \vee (p \wedge \sim q) \equiv p$.

ⓐ

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Equivalent

ⓑ

p	q	$p \vee q$	$\sim q$	$p \vee \sim q$	$(p \vee q) \wedge (p \vee \sim q)$
T	T	T	F	T	T
T	F	T	T	T	T
F	T	T	F	F	F
F	F	F	T	T	F

Equivalent

ⓒ

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \vee (p \wedge \sim q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	F	F	F
F	F	F	T	F	F

Equivalent

(b) Replacement process:

Consider a formula $A: P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula A.

If we replace $Q \to R$ by an equivalent formula $\sim Q \vee R$ in A. we get another formula $B: P \to (\sim Q \vee R)$.

One can easily verify that the formulas A & B are equivalent to each other.

This process of obtaining B from A is known as the replacement process.

Ex:- Prove that $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$. ✓

Sol:- Replacing $Q \to R$ by $\sim Q \vee R$. we get

$$P \to (Q \to R) = \underset{p}{P} \to \underset{q}{(\sim Q \vee R)}$$ which is equivalent to

$$= \sim P \vee (\sim Q \vee R)$$

$$= \sim P \vee \sim Q \vee R$$

$$= (\sim P \vee \sim Q) \vee R$$

$$= \sim (P \wedge Q) \vee R.$$

$$= (P \wedge Q) \to R.$$

Ex: Prove that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$.

Sol: $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\sim P \vee Q) \wedge (\sim R \vee Q)$

$\Leftrightarrow (\sim P \wedge \sim R) \vee Q$

$\left[\because (S_1 \vee S_2) \wedge (S_3 \vee S_2) \Leftrightarrow (S_1 \wedge S_3) \vee S_2\right]$

$\Leftrightarrow \sim (P \vee R) \vee Q$.

$\Leftrightarrow (P \vee R) \to Q$.

Ex: $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$. ✓

Sol: $P \to (Q \to P) \Leftrightarrow \sim P \vee (Q \to P)$.

$\Leftrightarrow \sim P \vee (\sim Q \vee P)$.

$\Leftrightarrow \sim P \vee \sim Q \vee P$.

$\Leftrightarrow (\sim P \vee P) \vee \sim Q$.

$\Leftrightarrow T \vee \sim Q$.

$\Leftrightarrow T$. (L.H.S).

$\sim P \to (P \to Q) \Leftrightarrow \sim (\sim P \vee (P \to Q))$.

$\Leftrightarrow \sim (\sim P \vee (\sim P \vee Q))$.

$\Leftrightarrow P \vee (\sim P \vee Q)$.

$\Leftrightarrow (P \vee \sim P) \vee Q$.

$\Leftrightarrow T \vee Q$

$\Leftrightarrow T$. (R.H.S.).

L.H.S = R.H.S.

so $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$.

Converse, contrapositive and inverse

we can arrange some new conditional statements using a conditional statements. $p \to q$.

Converse:-

Converse of the $p \to q$ is proposition $q \to p$.

Contrapositive:-

The contrapositive of $p \to q$ is the proposition $\sim q \to \sim p$.

inverse:-

The inverse of $p \to q$ is the proposition $\sim p \to \sim q$.

Note:- The above three conditional statements formed from $p \to q$.

Example:-

where p, q represents the statements.

p: Priya is concerned about her cholesterol levels.

q: priya walks at least two miles three times a week.

① **Implication** $(p \to q)$.

If priya is concerned about her cholesterol levels then she will walk at least two miles three times a week.

② **Converse** $(q \to p)$

If priya walks at least two miles three times a week then she is concerned about her cholesterol levels.

$\sim q \lor (\sim p \land q)$

$\sim p \land \sim q$

$\sim(p \lor q) \equiv p \lor q$

③ Contrapositive: $(\sim q \to \sim p)$

If priya weathers does not walks at least two miles three times a week then she is not concerned about her cholesterol levels.

④ Inverse: $(\sim p \to \sim q)$

If priya is not concerned about her cholesterol levels then she will not walks

Ex: consider the statements.

p: you are guilty   q: you are punished.

Ex: p: Today is a holiday.

q: I will go for a movie

Ex: let p, q and r be the propositions.

p: you have flu   q: you miss the final examination

r: you pass the course.

Express each of the following formula as an English sentence.

a) $p \to q$ b) $\sim q \leftrightarrow r$ c) $q \to \sim r$ d) $p \vee q \vee r$

e) $(p \to \sim r) \vee (q \to \sim r)$ f) $(p \wedge q) \vee (\sim q \vee r)$.

Ex: let p and q be the propositions:

p: I bought a lottery ticket this week.

q: I won the million dollar jackpot on friday.

Express each of the following formula as an English sentence:

a) $\sim p$ e) $p \vee q$

b) $p \to q$ f) $p \wedge q$

c) $p \leftrightarrow q$ g) $\sim p \to \sim q$

d) $\sim p \wedge \sim q$ h) $\sim p \vee (p \wedge q)$.

Functionally complete sets of connectives

(1) We have already defined the connectives $\wedge, \vee, \sim, \rightarrow$ and $\rightleftarrows$.

(2) Now introduce some other connectives namely NAND, NOR and XOR.

The word NAND is combination of NOT and AND
" NOR " " NOT and OR
" XOR " NOT and $\leftrightarrow$

$\therefore$ NAND = $\sim$ Conjunction $\sim\wedge$
NOR = $\sim$ disjunction $\sim\vee$
XOR = $\sim$ Biimplication. $\leftrightarrow$

(3) The connectives $\uparrow$ and $\downarrow$ have been defined in terms of the connectives $\wedge, \vee$ and $\sim$.

(4) Therefore, for any formula containing the connectives $\uparrow$ or $\downarrow$, one can obtain an equivalent formula containing the connectives $\wedge, \vee$ and $\sim$ only.

(5) Note:- $\uparrow$ and $\downarrow$ are duals of each other.

Therefore in order to obtain the dual of a formula which includes $\uparrow$ or $\downarrow$, we should interchange $\uparrow$ and $\downarrow$

Definition:- (functionally complete set of connectives)

If every formula can be expressed in terms of an equivalent formula containing the connectives only from this set.

(6) Functionally completed set should not contain any redundant connectives i.e

connective which can be expressed in terms of other connectives.

Already we have,

(1) $P \vee Q \Leftrightarrow \sim(\sim P \wedge \sim Q)$

(2) $P \wedge Q \Leftrightarrow \sim(\sim P \vee \sim Q)$.

(3) $P \rightarrow Q \Leftrightarrow \sim P \vee Q$.

(4) $P \leftrightarrow Q \Leftrightarrow (\sim P \vee Q) \wedge (P \vee \sim Q)$.

(7) Hence by first replacing all biconditionals, then the conditionals finally all the conjunctions or all the disjunctions in any formula.

(8) we can obtain an equivalent formula which contains either the negation and disjunction or only or the negation and conjunction only.

(9) In other words, for every formula, we can find an equivalent formula containing the connectives $\vee$ and $\sim$ or $\wedge$ and $\sim$ only.

(10) From the definition of functional complete set of connectives, the sets of connectives $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ are functionally ~~comp~~ complete sets.

(11) The set $\{\wedge, \vee\}$ is not functionally completed, as for the formula $\sim p$, it is not possible to find an equivalent formula containing connectives only from the set $\{\wedge, \vee\}$.

(12) $\{\uparrow\}$, $\{\downarrow\}$ are functionally complete.

<u>Proof</u>: In order to prove, it is sufficient to show that the sets of connectives $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ can be expressed either interms of $\uparrow$ alone or interms of $\downarrow$ alone.

To show that $\{\vee, \sim\}$ is functionally complete, it is enough to show that $\sim$ and $\vee$ can be expressed interms of $\downarrow$ alone.

We have

$$\sim p \Leftrightarrow \sim p \wedge \sim p \Leftrightarrow \sim(p \vee p) \Leftrightarrow p \downarrow p.$$

$$p \vee Q \Leftrightarrow \sim(\sim p \wedge \sim Q) \Leftrightarrow \sim p \downarrow \sim Q \Leftrightarrow \quad \sim(\sim p \vee \sim Q)$$

$$p \wedge Q.$$

$$\sim\sim(p \vee Q) \quad (\sim p \wedge \sim p) \downarrow (\sim Q \wedge \sim Q) \Rightarrow$$

$$\sim[p \downarrow Q] \quad \sim(p \vee p) \downarrow \sim(Q \vee Q)$$

$$\Rightarrow (p \downarrow p) \downarrow (Q \downarrow Q).$$

Then $\{\downarrow\}$ is a functionally completed set.

(13) To show the $\{\sim, \wedge\}$ is functionally completed, we have to express $\sim$ and $\wedge$ in terms of $\uparrow$ alone.

The following valid equalities help us in this direction.

$$\sim P \Leftrightarrow \sim P \vee \sim P \Leftrightarrow \sim(P \wedge P) \Leftrightarrow P \uparrow P.$$

$$P \wedge Q \Leftrightarrow \sim(\sim(P \wedge Q)) \qquad \sim(P\wedge Q) \uparrow \sim(P\wedge Q)$$

$$\Leftrightarrow \underline{\sim(P \uparrow Q)} \qquad (P\wedge Q) \wedge (P\wedge Q)$$

$$\Leftrightarrow [(P \uparrow Q) \uparrow (P \uparrow Q)]$$

Then $\{\uparrow\}$ is a functionally complete set.

$$(P\uparrow Q) \uparrow (P\uparrow Q)$$

$$\sim(P\wedge Q) \uparrow \sim(P\wedge Q)$$

$$\sim(\sim(P\wedge Q) \wedge \sim(P\wedge Q))$$

$$\sim \sim(P\wedge Q)$$

$$P\wedge Q \vee P\wedge Q$$

$$P\wedge Q.$$

$$P \vee Q = \sim(\sim(P \vee Q))$$

$$= \sim(P \downarrow Q)$$

$$(P\downarrow Q) \downarrow (P \downarrow Q) \qquad \sim(P\vee Q) \downarrow \sim(P \vee Q)$$

$$= \sim[\sim(P\vee Q) \vee \sim(P\vee Q)]$$

$$(P\vee Q) \wedge (P \vee Q)$$

$$\underline{\underline{P\vee Q}}$$

① Prove that $(P \to Q) \wedge (R \to Q) \Rightarrow (P \vee R) \to Q$. ✓

② Show that $(\sim P \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$.

③ Show that implications. ✓

ⓐ $\sim Q \wedge (P \to Q) \Rightarrow P$

$\sim Q \wedge (P \to Q) \to P$ is a T.

$\sim Q \wedge [\sim P \vee Q] \to P$.

$\sim [\sim Q \wedge (\sim P \vee Q)] \vee P$.

$[Q \vee \sim(\sim P \vee Q)] \vee P$

$[Q \vee (P \wedge \sim Q)] \vee P$.

$[(Q \vee P) \wedge (Q \vee \sim Q)] \vee P$.

$((P \vee Q) \wedge T) \vee P$

$P \vee Q \vee P \Leftrightarrow P \vee P \vee Q \Leftrightarrow P \vee Q$.

It is not a tautological implication.

ⓑ $\cancel{(P \vee Q) \wedge (\sim Q)} \Rightarrow Q$

$(P \vee Q) \wedge (\sim Q) \to Q$ is T

$\sim[(P \vee Q) \wedge (\sim Q)] \vee Q$.

$[\sim(P \vee Q) \vee \sim(\sim Q)] \vee Q$.

$(\sim(P \vee Q) \vee Q) \vee (Q)$.

$(\sim P \wedge \sim Q) \vee Q) \vee Q$.

$(\sim P \vee Q) \wedge (\sim Q \vee Q) \vee Q$

(b) $(P \vee Q) \wedge (\sim P) \Rightarrow Q$.

Sol: $((P \vee Q) \wedge (\sim P)) \rightarrow Q$. is T

$\sim((P \vee Q) \wedge (\sim P)) \vee Q$.

$(\sim(P \vee Q) \vee P) \vee Q$.

$((\sim P \wedge \sim Q) \vee P) \vee Q$.

$((\sim P \vee P) \wedge (\sim Q \vee P)) \vee Q$

$(T \wedge (P \vee \sim Q)) \vee Q$.

$(P \vee \sim Q) \vee Q$.

$P \vee Q \vee \sim Q \Rightarrow P \vee T \Rightarrow T$.

(c) $(P \rightarrow Q) \Rightarrow P \rightarrow (P \wedge Q)$.

$(P \rightarrow Q) \rightarrow (P \rightarrow (P \wedge Q))$ is T.

$(\sim P \vee Q) \rightarrow (P \rightarrow (P \wedge Q))$

$(\sim P \vee Q) \rightarrow (\sim P \vee (P \wedge Q))$

$(\sim P \vee Q) \rightarrow ((\sim P \vee P) \wedge (\sim P \vee Q))$

$(\sim P \vee Q) \rightarrow (T \wedge (\sim P \vee Q))$

$(\sim P \vee Q) \rightarrow (\sim P \vee Q)$.

$\sim(\sim P \vee Q) \vee (\sim P \vee Q)$

$(\sim P \vee Q) \vee \sim(\sim P \vee Q)$

$P \vee \sim P$

T.

$\therefore$ L.H.S = R.H.S $(P \vee Q \Leftrightarrow Q \vee P)$

Example: Prove that NAND ($\uparrow$) and NOR ($\downarrow$) both are commutative but not associative.

Ans:-

(i) $P \uparrow Q \Leftrightarrow Q \uparrow P$, $P \downarrow Q \Leftrightarrow Q \downarrow P$. (Commutative)

by using truth table

P	Q	$P \uparrow Q$	$Q \uparrow P$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

P	Q	$P \downarrow Q$	$Q \downarrow P$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

The connectives $\downarrow$, $\uparrow$ are commutative.

without constructing truth table

(i) $P \uparrow Q \Leftrightarrow \sim(P \wedge Q)$

$\Leftrightarrow \sim P \vee \sim Q$. — L.H.S

$Q \uparrow P \Leftrightarrow \sim(Q \wedge P)$

$\Leftrightarrow \sim Q \vee \sim P$

$\Leftrightarrow \sim P \vee \sim Q$ — R.H.S.

$\therefore$ L.H.S = R.H.S. $[\because (P \uparrow Q) \Leftrightarrow (Q \uparrow P)]$

(ii) $P \downarrow Q \Leftrightarrow Q \downarrow P$

$P \downarrow Q \Leftrightarrow \sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ — L.H.S

$Q \downarrow P \Leftrightarrow \sim(Q \vee P) \Leftrightarrow \sim Q \wedge \sim P$

$\Leftrightarrow \sim P \wedge \sim Q$. — R.H.S

$\therefore$ L.H.S = R.H.S $(P \downarrow Q \Leftrightarrow Q \downarrow P)$

(iii) The connectives ↑ and ↓ are not associative.

i.e $P \uparrow (Q \uparrow R) \not\Leftrightarrow (P \uparrow Q) \uparrow R$.

$P \downarrow (Q \downarrow R) \not\Leftrightarrow (P \downarrow Q) \downarrow R$.

L.H.S — $P \uparrow (Q \uparrow R) \Leftrightarrow P \uparrow \sim (Q \wedge R)$

$\Leftrightarrow \sim (P \wedge \sim (Q \wedge R))$.

$\Leftrightarrow \sim P \vee \sim(\sim(Q \wedge R))$

$\Leftrightarrow \sim P \vee (Q \wedge R)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee R)$

R.H.S — $(P \uparrow Q) \uparrow R \Leftrightarrow \sim(P \wedge Q) \uparrow R$

$\Leftrightarrow \sim (\sim (P \wedge Q) \wedge R)$.

$\Leftrightarrow (P \wedge Q) \vee (\sim R)$

$\Leftrightarrow (P \vee \sim R) \wedge (Q \vee \sim R)$.

$\therefore P \uparrow (Q \uparrow R) \not\Leftrightarrow (P \uparrow Q) \uparrow R$.

Similarly $P \downarrow (Q \downarrow R) \not\Leftrightarrow (P \downarrow Q) \downarrow R$.

By using truth table

P	Q	R	$P \uparrow Q$	$(P \uparrow Q) \uparrow R$ — ①	$Q \uparrow R$	$P \uparrow (Q \uparrow R)$ — ②
0	0	0	1	1	1	1
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	1
1	0	0	1	1	1	0
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	1	0	1

both are not equal.

Simillarly $(P \downarrow Q) \downarrow R \not\Leftrightarrow P \downarrow (Q \downarrow R)$.

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(ii) $(p \to q) \to q \Rightarrow (p \vee q)$

$(p \to q) \to q \to (p \vee q)$

$[\sim(p \to q) \vee q] \to (p \vee q)$

$[\sim(\sim p \vee q) \vee q] \to (p \vee q)$

$\sim[\sim(\sim p \vee q) \vee q] \vee (p \vee q)$

$[(\sim p \vee q) \wedge \sim q] \vee (p \vee q)$

$[(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \vee (p \vee q)$

$[(\sim p \wedge \sim q) \vee F] \vee (p \vee q)$

$(\sim p \wedge \sim q) \vee (p \vee q)$

$(\sim p \vee p \vee q) \wedge (\sim q \vee p \vee q)$

$(T \vee q) \wedge (T \vee p) \Rightarrow p \wedge q.$

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For any primitive statement p, q and r any tautology T_0 and any contradiction F_0.

Statement	meaning
① $\sim\sim p \Leftrightarrow p$	— Law of double negation
② $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$; $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$	— demorgan's Laws.
③ $p \vee q \Leftrightarrow q \vee p$; $p \wedge q \Leftrightarrow q \wedge p$	— commutative Laws.
④ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$; $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	— associative laws.
⑤ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$; $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	— distributive laws
⑥ $p \vee p \Leftrightarrow p$; $p \wedge p \Leftrightarrow p$	— Idempotent laws
⑦ $p \vee F_0 \Leftrightarrow p$, $p \wedge T_0 \Leftrightarrow p$	— Identity Laws.
⑧ $p \vee \sim p \Leftrightarrow T_0$; $p \wedge \sim p \Leftrightarrow F_0$	— Inverse Laws (or) Complement laws
⑨ $p \vee T_0 \Leftrightarrow T_0$; $p \wedge F_0 \Leftrightarrow F_0$	— domination laws
⑩ $p \vee (p \wedge q) \Leftrightarrow p$; $p \wedge (p \vee q) \Leftrightarrow p$	— Absorption Laws.

Statement	Meaning
(11) $p \to q \Leftrightarrow \sim p \vee q$	Law of Implication
(12) $\sim(p \to q) \Leftrightarrow p \wedge \sim q$	"
(13) $p \to q \Leftrightarrow \sim q \wedge \sim p$	Law of contrapositive
(14) $p \vee \sim p \Leftrightarrow T$; $p \wedge \sim p \Leftrightarrow F$	Complement law

---

Ex:- Pt $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$.

$\Rightarrow (\sim p) \wedge (\sim(\sim p \wedge q))$ — demorgan law

$\Rightarrow (\sim p) \wedge (\sim\sim p \vee \sim q)$ — ~~law of~~

$\Rightarrow \sim p \wedge (p \vee \sim q)$ — Law of double negation

$\Rightarrow (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ — distributive

$\Rightarrow F_0 \vee (\sim p \wedge \sim q)$ — By contradiction

$\Rightarrow \sim p \wedge \sim q = R.H.S$ By identity

$\therefore$ Hence $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$.

(2) $(p \wedge q) \to (p \vee q)$ Prove this is a tautology.

we know $p \to q \equiv \sim p \vee q$ (Law of Implication)

$\Rightarrow \sim(p \wedge q) \vee (p \vee q)$

$\Rightarrow \sim p \vee \sim q \vee p \vee q$ — demorgan's

$\Rightarrow (p \vee \sim p) \vee (q \vee \sim q)$ — Inverse Law

$\Rightarrow T \vee T$

$\Rightarrow T$.

(3) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Note: $p \leftrightarrow q = \overset{(p\wedge q)\vee(\sim p\wedge \sim q)}{\cancel{(p\wedge \sim q)\vee(q\wedge \sim p)}}$ or $(p\to q)\wedge(q\to p)$

$\sim(p \leftrightarrow q) \Rightarrow \sim(p\to q) \vee \sim(q \to p)$

$\Rightarrow \sim(\sim p \vee q) \vee \sim(\sim q \vee p)$

$\Rightarrow (\sim\sim p) \wedge \sim q \vee (\sim\sim q) \wedge \sim p$

$\Rightarrow p \wedge \sim q \vee q \wedge \sim p$

$\Rightarrow (p \wedge \sim p) \vee (q \wedge \sim q)$.

$\Rightarrow$ R.H.S.

(4)^{PT} $p \uparrow (q \uparrow r) \equiv \sim p \vee (q \wedge r)$ By using truth table.

(5) $(p \uparrow q) \uparrow r = (p \wedge q) \vee \sim r$

(6) $p \downarrow (q \downarrow r) \equiv \sim p \wedge (q \vee r)$.

(7) $(p \downarrow q) \downarrow r \equiv (p \vee q) \wedge \sim r$.

$\rightarrow (p \uparrow q) \uparrow r = [\sim(p\wedge q) \uparrow r]$

$= \sim(\sim(p \wedge q) \wedge r)$

$= (p \wedge q) \vee \sim r$.

$\rightarrow p \downarrow (q \downarrow r) = p \downarrow \sim(q \vee r)$

$= \sim(p \vee \sim(q \vee r))$

$= \sim p \wedge (q \vee r)$

Duality Law:-

Two formulas A and A^* are said to be the duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

→ The connectives $\wedge$ and $\vee$ are also called duals of each other.

→ If the formula A contains the special variable T or F then A^*, its dual is obtained by replacing T by F and F by T.

Ex:1 Write the duals of (i) $(P \wedge Q) \vee R$

Sol:- $(P \vee Q) \wedge R$

(ii) $(P \vee Q) \wedge F$

Sol:- $(P \wedge Q) \vee T$.

(iii) $\sim(P \wedge Q) \vee (P \wedge \sim(Q \vee \sim S))$.

Sol:- $\sim(P \vee Q) \wedge (P \vee \sim(Q \wedge \sim S))$.

~~Ex2: S.T~~ $\sim(\sim P \wedge \sim(Q \vee R)) \Leftrightarrow (P \vee (Q \vee R))$

~~A^*~~

Ex:1 $\sim(P \wedge Q) \rightarrow \sim P \vee (\sim P \vee Q)) \Leftrightarrow (\sim P \vee Q)$.

$\Leftrightarrow (P \wedge Q) \vee (\sim P \vee (\sim P \vee Q))$

$\Leftrightarrow (P \wedge Q) \vee (\sim P \vee Q)$

$\Leftrightarrow (P \vee \sim P) \wedge (P \vee Q) \wedge (Q \vee \sim P) \vee (Q \vee Q)$

$\Leftrightarrow Q \vee \sim P$

$\Leftrightarrow \sim P \vee Q$.

Normal forms

Normal forms:

To determine whether a given compound propositions $A(P_1, P_2 \cdots P_n)$ is a tautology or a contradiction or atleast satisfiable. and whether two given compound propositions $A(P_1, P_2 \cdots P_n)$ and $B(P_1, P_2 \cdots P_n)$ are equivalent, we have to construct the truth tables and compare them.

→ But the construction of truth tables may not be practical, when the no: of primary propositions (variables) $P_1, P_2 \cdots P_n$ increases.

→ A Better method is to reduce A and B to some standard forms called "normal forms". (or) "Canonical form"

And use them for deciding the nature of A & B and for comparing A and B.

There are two types of normal forms.

① DNF (disjunctive normal form) SOP

② CNF (conjunctive normal form) POS

Note:- we shall use the word 'product' in place of conjunction 'sum' in place of disjunction.

① DNF: (disjunction normal form)

→ A ~~product~~ ~~sum~~ product of the variables and their negations are called an elementary ~~product~~ ~~sum~~ product

i.e A formula which is equivalent to a given formula and which consists of a ~~product~~ ~~sum~~ sum of elementary ~~products~~ ~~sum~~ product is called disjunctive normal form.(DNF).

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Ex: Elementary sum: q, $\sim q$, $p \wedge \sim q$, $\sim p \wedge \sim q$.

(5) Ex: $p \wedge (p \rightarrow q)$ write DNF

Sol: $p \wedge (\sim p \vee q)$.

$\Rightarrow \underbrace{(p \wedge \sim p)}_{\text{product}} \underbrace{\vee}_{\text{sum}} (p \wedge q)$ — sum of product.

$\therefore$ this is DNF.

DNF

$\wedge$ $\vee$

$*$ $+$

$(\sim p \wedge \sim q) \vee (\sim p \wedge p)$; $(\sim q) \vee (q \wedge q)$

(6) $\sim (p \vee q) \longleftrightarrow (p \wedge q)$. write DNF.

$(p \rightarrow q) \wedge (q \rightarrow p)$

$(\sim p \vee q) \wedge (\sim q \vee p)$

$(\sim p \wedge \sim q) \vee q \wedge \sim q \vee \sim p p$

$\boxed{p \leftrightarrow q :- (p \wedge q) \vee (\sim p \wedge \sim q)}$.

Sol:- $[\sim(p \vee q) \wedge (p \wedge q)] \vee [\sim(\sim(p \vee q)) \wedge \sim(p \wedge q)]$

$\Leftrightarrow [\sim p \wedge \sim q \wedge p \wedge q] \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$

$\Leftrightarrow [\sim p \wedge \sim q \wedge p \wedge q] \vee [[(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]]$

$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \wedge \sim p) \vee (q \wedge \sim p)] \vee$
$[(p \wedge \sim q) \vee (q \wedge \sim q)]$

$\Leftrightarrow \underbrace{(\sim p \wedge \sim q \wedge p \wedge q)}_{\text{product}} \underbrace{\vee}_{\text{sum}} (p \wedge \sim p) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \vee$
$(q \wedge \sim p)$.

$\therefore$ sum of product.

Ex:- $(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$ write DNF.

$(p \rightarrow q = \sim p \vee q)$

$\Leftrightarrow$ $(p \rightarrow \sim q) \wedge (\sim q \rightarrow p)$

$(\sim p \vee \sim q) \rightarrow (~~(p \wedge \sim q) \vee (\sim p \wedge q)~~)$.

$\Leftrightarrow \sim(\sim p \vee \sim q) \vee [(\sim p \vee \sim q) \wedge (q \vee p)]$

$(x+y)\cdot(x'+y')$

$\Leftrightarrow (p \wedge q) \vee \sim(p \wedge q)$

$xx' + xy' + yx' + yy'$

$\Leftrightarrow (p \wedge q) \vee [(\sim p \wedge q) \vee \underbrace{(\sim p \wedge p)}_{F_0} \vee \underbrace{(\sim q \wedge q)}_{F_0} \vee (\sim q \wedge p)]$

$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee F_0 \vee F_0 \vee (p \wedge \sim q)$

$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$. Sum of product.

$\therefore$ this is DNF.

Procedure of obtain the DNF or CNF

① If the connectives $\rightarrow$ and $\leftrightarrow$ are present in a given formula, they are replaced by $\wedge$, $\vee$ and $\sim$.

② If the negation is present before a given formula or a part of the given formula, De morgan Laws are applied so that negation is brought before the variables only.

③ If necessary, the distributive and idempotent laws are applied.

Ex:

CNF :- (conjunctive normal form)

A ~~product~~ sum of variables and their negations are called an elementary ~~product~~ sum

i.e

A formula which is equivalent to a given formula and consist of product of elementary sum is called "conjunctive normal form" of the given formula.

Elementary ~~product~~ sum : $p, \sim p, p \vee \sim p, \sim p \vee q, \sim p \vee \sim q$

Ex:- $p \wedge (p \to q)$ write CNF.

$$p \wedge (p \to q) \Leftrightarrow \underbrace{p}_{\text{product}} \wedge \underbrace{(\sim p \vee q)}_{\text{sum}} \to \text{product of sum.}$$

∴ this is required form.

$p \leftrightarrow q$

$(p \to q) \wedge (q \to p)$

Ex:- $\sim(p \vee q) \leftrightarrow (p \wedge q)$ write CNF.

$p \to q$

$\Rightarrow \sim p \vee q$

$$\Leftrightarrow [(\sim(p \vee q) \to (p \wedge q)] \wedge [(p \wedge q) \to \sim(p \vee q)]$$

$$\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee \sim(p \vee q)]$$

$$\Leftrightarrow [(p \vee q \vee p) \wedge (p \vee q \vee q)] \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$$

$$\Leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$$

$$\Leftrightarrow (p \vee q) \wedge (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim q).$$

$p \vee p : p$

$p \wedge p : p$

product of sum.

$+ \quad \vee$

$* \quad \wedge$

∴ this is CNF.

Ex:- $p \wedge (p \to q)$ write CNF.

$\Leftrightarrow p \wedge (\sim p \vee q)$.

$\Leftrightarrow (p \vee F) \wedge (\sim p \vee q)$

$p \vee F_0 \Rightarrow p$

$q \wedge \sim q = F_0$

$\Leftrightarrow (p \vee (q \wedge \sim q)) \wedge (\sim p \vee q)$

$\Leftrightarrow (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$.

$\therefore$ this is product of sum.

Principal disjunctive normal form (PDNF).

→ For a given formula, an equivalent formula consisting of disjunction of minterms is known as its PDNF.

→ This is also called "sum of products canonical form". (Sum of minterms)

Ex: let $P_1, P_2 \cdots P_n$ be n statement variables.

the expression $P_1^* \wedge P_2^* \wedge P_3^* \cdots \wedge P_n^*$.

where P_i^* is either P_i or $\sim P_i$ called minterms.

There are 2^n such minterms for n variables.

Ex:

p	q	r	minterms
0	0	0	$\sim p \wedge \sim q \wedge \sim r$
0	0	1	$\sim p \wedge \sim q \wedge r$
0	1	0	$\sim p \wedge q \wedge \sim r$
0	1	1	$\sim p \wedge q \wedge r$
1	0	0	$p \wedge \sim q \wedge \sim r$
1	0	1	$p \wedge \sim q \wedge r$
1	1	0	$p \wedge q \wedge \sim r$
1	1	1	$p \wedge q \wedge r$.

$F \to \sim p$

$F \to p$

Minterms:

Given a no: of variables, the products in which each variable or its negation, but not both, occurs only once are called minterms.

~~Definition~~: **Note**:- there are 2^n minterms for n variables.

Ex: for the two variables P and Q, the possible 4 minterms are

$P \wedge Q$, $P \wedge \sim Q$, $\sim P \wedge Q$ and $\sim P \wedge \sim Q$.

PDNF:- sum of minterms - whose truth value is 'T'.

PCNF: (" maxterms)! - " " "F".

= product of maxterms.

Principal conjunctive normal forms: (PCNF)

For a given formula, an equivalent formula consisting of conjunction of the maxterms only is known as principal conjunctive normal form (PCNF).

let $P_1, P_2, P_3 \ldots P_n$ be n variable statements.

then the expression $P_1^* \vee P_2^* \vee P_3^* \ldots P_n^*$

where P_i or $\sim P_i$ is called maxterms.

Note:- There are 2^n maxterms for n variables.

maxterms:-

Given a no: of variables, the sums in which each variable or its negation, but not both, occurs only once is called the maxterms.

Ex: for two variables P and Q, the possible 4 ~~maxterms~~ maxterms are

$P \vee Q$, $P \vee \sim Q$, $\sim P \vee Q$, and $\sim P \vee \sim Q$.

For 3 variables possible maxterms are

Ex)

p	q	r	maxterms
0	0	0	$p \vee q \vee r$
0	0	1	$p \vee q \vee \sim r$
0	1	0	$p \vee \sim q \vee r$
0	1	1	$p \vee \sim q \vee \sim r$
1	0	0	$\sim p \vee q \vee r$
1	0	1	$\sim p \vee q \vee \sim r$
1	1	0	$\sim p \vee \sim q \vee r$
1	1	1	$\sim p \vee \sim q \vee \sim r$

p q

F F F $F \wedge F \wedge F$

$\sim p \wedge \sim q \wedge \sim r$

$\sim p \wedge q \wedge \sim r$

$\sim p \vee \sim q \vee \sim r$

Note:- maxterms are simply duals of minterms.

Ex:- By using truth table construct PDNF & PCNF for $p \to q$, $p \vee q$ and $\sim p \wedge q$.

p	q	$p \to q$	$p \vee q$	$\sim p \wedge q$	$\sim p \wedge q$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	T	F

PDNF $\to$ T.

$p \to q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$. $(\sim p \vee \sim q)$

$(p \vee q) \Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$.

$(\sim p \wedge q) \Leftrightarrow (\sim p \wedge q)$.

PCNF $\to$ F

$(p \vee q) \Leftrightarrow (p \vee q)$.

$(p \to q) \Leftrightarrow (\sim p \vee q)$, $(\sim p \wedge q) \Leftrightarrow (\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$.

Ex: Write PDNF for $(\sim p \vee q) \Leftrightarrow (\cancel{\sim p \wedge (q \vee \sim q)}) \vee$

$(\cancel{q \wedge (p \vee \sim p)})$.

Ex:

Sol: $p \vee \sim p \Leftrightarrow T_0$ (Inverse law) $p \wedge \sim p = F$ product $(\wedge) \to$ conjunction

$q \wedge T_0 \Leftrightarrow q$ (identity law). $q \vee F = q$ sum $(\vee) \to$ disjunction.

$(\sim p \vee q) \Leftrightarrow (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$

$\Rightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

distributive law.

$\Rightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q)$

commutative law.

$\Rightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$. $(p \vee p = p)$ idempotent law

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Ex: Write PDNF for $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$.

$[(P \wedge Q) \wedge (R \vee \sim R)] \vee [(\sim P \wedge R) \wedge (Q \vee \sim Q)] \vee [Q \wedge R \wedge (P \vee \sim P)]$

$\boxed{\begin{array}{l} P \vee \sim P = T_0 \\ Q \wedge T_0 = Q \end{array}}$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee$

$(Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$.

$\boxed{P \vee P = P}$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R)$.

$\therefore$ this is PDNF

Ex: $P \vee (P \wedge Q) \Leftrightarrow P$.

$P \vee (P \wedge Q) \Rightarrow P \wedge \underset{T_0}{\underline{(Q \vee \sim Q)}} \vee (P \wedge Q)$

$\Rightarrow \underline{(P \wedge Q)} \vee (P \wedge \sim Q) \vee \underline{(P \wedge Q)}$.

$\Rightarrow (P \wedge Q) \vee (P \wedge \sim Q) = L.H.S$ $\boxed{P \vee P = P}$

$P \Leftrightarrow P \wedge (Q \vee \sim Q)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) . = R.H.S$

$\therefore L.H.S = R.H.S.$

Ex: $P \vee (\sim P \wedge Q) \Leftrightarrow P \vee Q$.

$P \vee (\sim P \wedge Q) \Rightarrow P \wedge (Q \vee \sim Q) \vee (\sim P \wedge Q)$

$\Rightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q) . = L.H.S.$

$P \vee Q \Rightarrow (P \wedge (Q \vee \sim Q)) \vee (Q \wedge (P \vee \sim P))$

$\Rightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (Q \wedge P) \vee (Q \wedge \sim P)$.

$\Rightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q) = R.H.S$

$\therefore L.H.S = R.H.S.$

Ex: $P \rightarrow ((P \rightarrow Q) \wedge \sim (\sim Q \vee \sim P))$ write PDNF.

$P \rightarrow ((P \rightarrow Q) \wedge \sim (\sim Q \vee \sim P)) \Leftrightarrow P \rightarrow ((\sim P \vee Q) \wedge (P \wedge Q))$

$\Leftrightarrow \sim P \vee [(\sim P \vee Q) \wedge (P \wedge Q)]$

$\Leftrightarrow \sim P \vee [\sim P \wedge (P \wedge Q) \vee Q \wedge (P \wedge Q)]$

$\Leftrightarrow \sim P \vee ((\sim P \wedge P) \wedge (\sim P \wedge Q) \vee \underline{[(Q \wedge P) \wedge (P \wedge Q)]}$

$\Leftrightarrow \sim P \vee [(F_0 \wedge \sim P \wedge Q) \vee \underline{(P \wedge Q)}]$ $P \wedge Q \wedge Q \wedge Q$.

$\therefore$ PDN

$\Leftrightarrow \sim P \vee (((F_0 \wedge P) \wedge Q) \vee (P \wedge Q))$

$\Leftrightarrow \sim P \vee [(F_0 \wedge Q) \vee (P \wedge Q)]$

$\Leftrightarrow \sim P \vee (F_0 \vee (P \wedge Q))$

$\Leftrightarrow \sim P \vee (P \wedge Q)$

$\Leftrightarrow [\sim P \wedge (Q \vee \sim Q)] \vee (P \wedge Q)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q).$

$Q \wedge Q = Q$
$P \wedge \sim P = F_0$
$P \wedge F_0 = F_0$
$P \vee F_0 = P$

$\therefore$ this is required PDNF.

PCNF

Ex: obtain PCNF of the formula 'S' given by $(\sim P \to R) \wedge (Q \leftrightarrow P)$.

$\Leftrightarrow (\sim P \to R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow (\sim P \to R) \wedge (Q \to P) \wedge (P \to Q).$

$\Leftrightarrow (P \vee R) \wedge (\sim Q \vee P) \wedge (\sim P \vee Q).$

$\Leftrightarrow P \vee R \vee \underbrace{(Q \wedge \sim Q)}_{F_0} \wedge (\sim Q \vee P \vee (R \wedge \sim R)) \wedge (\sim P \vee Q \vee (R \wedge \sim R))$

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \sim Q) \wedge (\sim Q \vee P \vee R) \wedge (\sim Q \vee P \vee \sim R)$
$\wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R).$

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R)$
$\wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R).$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge$
$(\sim P \vee Q \vee \sim R).$

$\therefore$ this required PCNF.

PDNF is negation of PCNF.

$S = PCNF$

$\sim S = \sim PDNF$.

$\Leftrightarrow \sim[(P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R)]$.

$\Leftrightarrow \sim(P \vee Q \vee R) \vee \sim(P \vee \sim Q \vee R) \vee \sim(P \vee \sim Q \vee \sim R) \vee \sim(\sim P \vee Q \vee R) \vee \sim(\sim P \vee Q \vee \sim R)$

$\Leftrightarrow (\sim P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R) \wedge (\sim P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee (P \wedge \sim Q \wedge R)$.

Ex: Obtain the PCNF for $(Q \to P) \wedge (\sim P \wedge Q)$.

$S: (Q \to P) \wedge (\sim P \wedge Q)$

$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \wedge Q)$

$\Leftrightarrow (\sim Q \vee P) \wedge [[\sim P \vee (Q \wedge \sim Q)] \wedge (Q \vee (P \wedge \sim P))]$

$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P) \wedge (Q \vee \sim P)$

$\boxed{P \vee P = P}$

$\Leftrightarrow (P \vee \sim Q) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (P \vee Q)$.

Ex: Obtain the principal disjunctive and conjunctive normal forms of

Sol: $P \to ((P \to Q) \wedge \sim(\sim P \vee \sim Q))$.

$\Leftrightarrow \sim P \vee ((P \to Q) \wedge \sim(\sim P \vee \sim Q))$

$\Leftrightarrow \sim P \vee ((\sim P \vee Q) \wedge (P \wedge Q))$

$\Leftrightarrow [\sim P \wedge (Q \vee \sim Q)] \vee [[\sim P \wedge (Q \vee \sim Q)] \vee (Q \wedge (P \vee \sim P)) \vee (Q \wedge P)]$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (Q \wedge P) \vee (Q \wedge \sim P) \vee (P \wedge Q)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q)$. is the PDNF.

$\sim S \Leftrightarrow$ remaining terms in PDNF.

$\Leftrightarrow (P \wedge \sim Q)$.

PCNF $\Leftrightarrow \sim(\sim S) \Leftrightarrow \sim(P \wedge \sim Q)$

$\Leftrightarrow \sim P \vee Q$ is the PCNF.

Ex: obtain the PDNF of $P \wedge (P \to Q)$.

P	Q	$P \to Q$	$P \wedge (P \to Q)$	minterm
T	T	T	T	$P \wedge Q$ (PDNF)
T	F	F	F	$\sim P \vee Q$
F	T	T	F	$P \vee \sim Q$
F	F	T	F	$P \vee Q$

$\left.\begin{matrix} \sim P \vee Q \\ P \vee \sim Q \\ P \vee Q \end{matrix}\right\}$ PCNF.

Ex: obtain the PDNF of $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$.
by using truth table.

Sol:- minterms: $P \wedge Q \wedge R$, $P \wedge Q \wedge \sim R$, $\sim P \wedge Q \wedge R$, $\sim P \wedge \sim Q \wedge R$.

PDNF = sum of minterms.

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R)$

Ex: obtain the PCNF of $\underline{(\sim P \to R) \wedge (Q \leftrightarrow P)}$ by using truth table & replacement process.

Sol:- minterm $\Rightarrow P \wedge \sim Q \wedge R$, $\sim P \wedge Q \wedge R$, $P \wedge \sim Q \wedge \sim R$,
$\sim P \wedge Q \wedge \sim R$, $\sim P \wedge \sim Q \wedge \sim R$.

PCNF $= (\text{sum of the minterm})'$

$= [(P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge \sim R)$
$\vee (\sim P \wedge Q \wedge \sim R)]'$

$\Rightarrow (\sim P \vee Q \vee \sim R) \wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge (P \vee Q \vee R) \wedge$
$(P \vee \sim Q \vee R)$.

Example : Prove the following

① $p \to (q \to p) \Leftrightarrow p \to (p \to q)$.

L.H.S. $p \to (q \to p) \Leftrightarrow p \to (\sim q \vee p)$

$\Leftrightarrow \sim p \vee \sim q \vee p$

$\Leftrightarrow \sim p \vee p \vee \sim q$

$\Leftrightarrow p \vee \sim p \vee \sim q$

$\Leftrightarrow T \vee \sim q \Leftrightarrow T$.

R.H.S $p \to (p \to q) \Leftrightarrow p \to (\sim p \vee q)$.

$\Leftrightarrow \sim p \vee (\sim p \vee q)$.

$\Leftrightarrow \sim p \vee \sim p \vee q$.

$\Leftrightarrow \sim p \vee q$.

p	q	$q \to p$	$p \to (q \to p)$	$p \to q$	$p \to (p \to q)$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	T	F	F
T	T	T	T	T	T

not logical equivalent.

(2) $p \rightarrow (q \vee r) \Leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$.

L.H.S $p \rightarrow (q \vee r) \Leftrightarrow \sim p \vee (q \vee r)$.

$\Leftrightarrow \sim p \vee q \vee r$.

$\Leftrightarrow$.

R.H.S $(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow (\sim p \vee q) \vee (\sim p \vee r)$.

$\Leftrightarrow \sim p \vee q \vee \sim p \vee r$.

$\Leftrightarrow (\sim p \vee \sim p) \vee q \vee r$.

$\Leftrightarrow \sim p \vee q \vee r$.

L.H.S = R.H.S.

$\therefore$ Both are logically equivalent.

(3) $(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (p \vee r) \rightarrow q$.

L.H.S $(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (\sim p \vee q) \wedge (\sim r \vee q)$.

$\Leftrightarrow (\sim p \wedge \sim r) \vee q$.

$\Leftrightarrow q \vee (\sim p \wedge \sim r)$.

$\Leftrightarrow q \vee \sim (p \vee r)$.

$\Leftrightarrow \sim (p \vee r) \wedge q$.

$\Leftrightarrow (p \vee r) \rightarrow q$.

④ $\sim(p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge \sim(p \wedge q)$.

L.H.S $\sim(p \leftrightarrow q) \Leftrightarrow \sim[(p \to q) \wedge (q \to p)]$

$\Leftrightarrow \sim((\sim p \vee q) \wedge (\sim q \vee p))$.

$\Leftrightarrow \sim(\sim p \vee q) \vee \sim(\sim q \vee p)$

$\Leftrightarrow (p \wedge \sim q) \vee (q \wedge \sim p)$.

R.H.S $(p \vee q) \wedge \sim(p \wedge q) \Leftrightarrow \underbrace{(p \vee q)}_{a} \wedge \sim(p \wedge q)$ $(\sim p \vee \sim q)$.

$\Leftrightarrow ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$

$\Leftrightarrow ((\sim p \wedge p) \vee (\sim p \wedge q)) \vee ((p \wedge \sim q) \vee (q \wedge \sim q))$

$\Leftrightarrow F \vee (\sim p \wedge q) \vee ((p \wedge \sim q) \vee F)$

$\Leftrightarrow (\sim p \wedge q) \vee (p \wedge \sim q)$.

Tautological Implications.

1) prove that $(P \to Q) \Rightarrow (\sim Q \to \sim P)$.

2) prove that $\sim Q \wedge (P \to Q) \Rightarrow \sim P$.

3) ST $\sim (P \to Q) \Rightarrow P$.

Show that following implications

1) $(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$.

2) $Q \Rightarrow P \to R$.

3) $(P \wedge Q) \Rightarrow P \to Q$.

Show the following implications without constructing the truth tables.

1) $\sim Q \wedge (P \to Q) \Rightarrow \sim P$.

b) $(P \vee Q) \wedge (\sim P) \Rightarrow Q$.

c) $(P \to Q) \Rightarrow P \to (P \wedge Q)$.

d) $(P \to Q) \to Q \Rightarrow P \vee Q$.

e) $((P \vee \sim P) \to Q) \to ((P \vee \sim P) \to R) \Rightarrow (Q \to R)$.

f) $(Q \to (P \wedge \sim P)) \to (R \to (R \wedge \sim P)) \Rightarrow (R \to Q)$

Tautological Implications without constructing truth tables.

1) $\sim Q \wedge (P \to Q) \to \sim P$.

$\sim Q \wedge (\sim P \vee Q) \to \sim P$

$(\sim Q \wedge \sim P) \vee (\sim Q \wedge Q) \to \sim P$

$\sim [(\sim Q \wedge \sim P) \cancel{\wedge} (\sim Q \wedge Q)] \vee \sim P$.

$(P \vee Q) \wedge \underline{(Q \vee \sim Q)} \vee \sim P$.

$(P \vee Q) \wedge \overset{T}{(T \vee \sim P)}$

$(P \vee Q) \wedge T$

$\sim [(\sim Q \wedge \sim P) \vee F] \vee \sim P$

$\sim (\sim Q \wedge \sim P) \vee \sim P$.

$P \vee Q \vee \sim P$.

$T \vee Q = T$.

2) $(P \vee Q) \wedge \sim P \Rightarrow Q$.

$((P \vee Q) \wedge \sim P) \to Q$

$\sim ((P \vee Q) \wedge \sim P) \vee Q$.

$((\sim P \wedge \sim Q) \vee P) \vee Q$.

$((P \vee \sim P) \wedge (P \vee \sim P)) \vee Q$.

$T \wedge T \vee Q$.

$T \vee Q$

$= T$.

③ $(P \to Q) \Rightarrow (P \to (P \wedge Q))$

$(P \to Q) \to (P \to (P \wedge Q))$.

$(\sim P \vee Q) \to (\sim P \vee (P \wedge Q))$.

$(\sim P \vee Q) \to ((\sim P \vee P) \wedge (\sim P \vee Q))$

$\sim(\sim P \vee Q) \vee (T \wedge (\sim P \vee Q))$

$(P \wedge \sim Q) \vee (\sim P \vee Q)$.

$(P \vee \sim P \vee Q) \wedge (\sim Q \vee \sim P \vee Q)$.

$T \wedge (T \vee \sim P)$

$T \wedge T \Rightarrow T$.

④ $(P \to Q) \to Q \Rightarrow (P \vee Q)$.

$((P \to Q) \to Q) \to (P \vee Q)$.

$((\sim P \vee Q) \to Q) \to (P \vee Q)$.

$(\sim(\sim P \vee Q) \vee Q) \vee (P \vee Q)$.

$((P \wedge \sim Q) \vee Q) \vee (P \vee Q)$.

$((P \vee Q) \wedge (\sim Q \vee Q)) \vee (P \vee Q)$.

$((P \vee Q) \wedge T) \vee (P \vee Q)$.

$F \vee (P \vee Q)$.

$\underline{P \vee Q}$.

not Totological implies.

⑤ $((P \vee \sim P) \to Q) \to ((P \vee \sim P) \to R) \Rightarrow Q \to R$.

$(\sim(P \vee \sim P) \vee Q) \to (\sim(P \vee \sim P) \vee R) \to (Q \to R)$.

$\sim((\sim P \wedge P) \vee Q) \vee ((\sim P \wedge P) \vee R) \to (\sim Q \vee R)$.

$((P \vee \sim P) \wedge \sim Q) \vee (F \vee R) \to (\sim Q \vee R)$.

$(F \wedge \sim Q) \vee QR \to (\sim Q \vee R)$

$\sim Q \vee R \to (\sim Q \vee R)$.

$\sim(\sim Q \vee R) \vee (\sim Q \vee R)$.

T.

⑥ $(Q \to (P \wedge \sim P)) \to (R \to (P \wedge \sim P)) \Rightarrow (R \to Q)$.

$[(\sim Q \vee (P \wedge \sim P)) \to (\sim R \vee (P \wedge \sim P))] \to R(\sim R \vee Q)$.

$[(\sim Q \vee P) \wedge (\sim Q \vee \sim P)] \to [(\sim R \vee P) \wedge (\sim R \vee \sim P)] \to (\sim R \vee Q)$.

$\sim((\sim Q \vee P) \wedge (\sim P \vee \sim Q)) \vee ((P \vee \sim R) \wedge (\sim P \vee \sim R)) \to (\sim R \vee Q)$.

$(\sim P \wedge Q) \vee (P \wedge Q) \vee [(P \vee \sim R) \wedge (\sim P \vee \sim R)] \to (\sim R \vee Q)$

$\sim((\sim P \wedge Q) \vee (P \wedge Q) \vee ((P \vee \sim R) \wedge (\sim P \vee \sim R))) \vee (Q \vee \sim R)$

$((P \vee \sim Q) \wedge (\sim P \vee \sim Q)) \wedge ((\sim P \wedge R) \vee (P \wedge R)) \vee (Q \vee \sim R)$

$(\cancel{(P \vee \sim Q) \wedge} (\sim P)) \vee$

Write DNF

① $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

$(\sim(p \vee q) \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow \sim(p \vee q))$

$((p \vee q) \vee (p \wedge q)) \wedge (\sim(p \wedge q) \vee \sim(p \vee q))$

$(p \vee q \vee p) \wedge (p \vee q \vee q) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$

$(p \vee q) \wedge (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$

$(p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$.

$(p \vee q) \wedge \underset{Q}{\underline{(\sim p \vee \sim q)}}$.

$(p \wedge (\sim p \vee \sim q)) \vee (q \wedge (\sim p \vee \sim q))$.

$(p \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim p) \vee (q \wedge \sim q)$

Find CNF

① $(\sim p \rightarrow q) \wedge (p \leftrightarrow q)$

② obtain DNF.

$Q \vee (P \wedge R) \wedge \sim((P \vee R) \wedge Q)$.

1) Obtain PDNF of the following formulas.

a) $P \vee (\sim P \wedge \sim Q \wedge R)$

b) $(Q \wedge \sim R \wedge \sim S) \vee (R \wedge S)$.

CNF

① $p \wedge (p \rightarrow q)$ write CNF.

$p \wedge (\sim p \vee q)$ — product of sum.

② $\sim(p \vee q) \longleftrightarrow (p \wedge q)$ write CNF.

$[\sim(p \vee q) \rightarrow (p \wedge q)] \wedge [(p \wedge q) \rightarrow \sim(p \vee q)]$

$[\cancel{\sim}\cancel{\sim}(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee \sim(p \vee q)]$

$(p \vee q) \vee (p \wedge q) \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$

$[(p \vee q \vee p) \wedge (p \vee q \vee q)] \wedge [(\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)]$

$(p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$.

$\Leftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$

↓ product ↓ sum

∴ This is required form

Ex: write PDNF for $(\sim p \vee q)$

identity law:

$p \wedge T =$

$(\sim p \wedge T) \vee (q \wedge T)$.

$[\sim p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)]$

$[(\sim p \wedge q) \vee (\sim p \wedge \sim q)] \vee [(q \wedge p) \vee (q \wedge \sim p)]$

$(\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$.

PCNF:

$(\sim p \to r) \land (q \leftrightarrow p)$.

$(\sim p \lor r) \land [(q \to p) \land (p \to q)]$

$((\sim p \lor r) \lor F) \land (\sim q \lor p \lor F) \land (\sim p \lor q \lor F)$.

$[\sim p \lor r \lor (q \land \sim q)] \land [\sim q \lor p \lor (r \land \sim r)] \land$
$[(\sim p \lor q) \lor (r \land \sim r)]$

$(\sim p \lor r \lor q) \land (\sim p \lor r \lor \sim q) \land (\sim q \lor p \lor r) \land$
$(\sim q \lor p \lor \sim r) \land (\sim p \lor q \lor r) \land$
$(\sim p \lor q \lor \sim r)$.

Unit – I

Mathematical Logic

INTRODUCTION

Proposition: A **proposition** or **statement** is a declarative sentence which is either true or false but not both. The truth or falsity of a proposition is called its **truth-value.**

These two values 'true' and 'false' are denoted by the symbols T and F respectively. Sometimes these are also denoted by the symbols 1 and 0 respectively.

Example 1: Consider the following sentences:

1. Delhi is the capital of India.
2. Kolkata is a country.
3. 5 is a prime number.
4. 2 + 3 = 4.

These are propositions (or statements) because they are either true of false.

Next consider the following sentences:

5. How beautiful are you?
6. Wish you a happy new year
7. $x + y = z$
8. Take one book.

These are not propositions as they are not declarative in nature, that is, they do not declare a definite truth value T or F.

Propositional Calculus is also known as **statement calculus.** It is the branch of mathematics that is used to describe a logical system or structure. A logical system consists of (1) a universe of propositions, (2) truth tables (as axioms) for the logical operators and (3) definitions that explain equivalence and implication of propositions.

Connectives

The words or phrases or symbols which are used to make a proposition by two or more propositions are called **logical connectives** or **simply connectives.** There are five basic connectives called negation, conjunction, disjunction, conditional and biconditional.

Negation

The **negation** of a statement is generally formed by writing the word 'not' at a proper place in the statement (proposition) or by prefixing the statement with the phrase 'It is not the case that'. If p denotes a statement then the negation of p is written as p and read as 'not p'. If the truth value of p is T then the truth value of p is F. Also if the truth value of p is F then the truth value of p is T.

Table 1. Truth table for negation

p	¬p
T	F
F	T

Example 2: Consider the statement p: Kolkata is a city. Then $\neg p$: Kolkata is not a city.

Although the two statements 'Kolkata is not a city' and 'It is not the case that Kolkata is a city' are not identical, we have translated both of them by p. The reason is that both these statements have the same meaning.

Conjunction

The **conjunction** of two statements (or propositions) p and q is the statement $p \wedge q$ which is read as 'p and q'. The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T. Otherwise it has truth value F.

Table 2. Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3: Consider the following statements p : It is raining today.

q : There are 10 chairs in the room.

Then $p \wedge q$: It is raining today and there are 10 chairs in the room.

Note: Usually, in our everyday language the conjunction 'and' is used between two statements which have some kind of relation. Thus a statement 'It is raining today and $1 + 1 = 2$' sounds odd, but in logic it is a perfectly acceptable statement formed from the statements 'It is raining today' and '$1 + 1 = 2$'.

Example 4: Translate the following statement:

'Jack and Jill went up the hill' into symbolic form using conjunction.

Solution: Let p : Jack went up the hill, q : Jill went up the hill.

Then the given statement can be written in symbolic form as $p \wedge q$.

Disjunction

The **disjunction** of two statements p and q is the statement $p \vee q$ which is read as 'p or q'. The statement $p \vee q$ has the truth value F only when both p and q have the truth value F. Otherwise it has truth value T.

Table 3: Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5: Consider the following statements p : I shall go to the game.

q : I shall watch the game on television.

Then $p \vee q$: I shall go to the game or watch the game on television.

Conditional proposition

If p and q are any two statements (or propositions) then the statement $p \rightarrow q$ which is read as, 'If p, then q' is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective.**

The conditional is defined by the following table:

Table 4. Truth table for conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In this conditional statement, p is called the **hypothesis** or **premise** or **antecedent** and q is called the **consequence** or **conclusion.**

To understand better, this connective can be looked as a conditional promise. If the promise is violated (broken), the conditional (implication) is false. Otherwise it is true. For this reason, the only circumstances under which the conditional $p \rightarrow q$ is false is when p is true and q is false.

Example 6: *Translate the following statement:*

'The crop will be destroyed if there is a flood' into symbolic form using conditional connective.

Solution: Let c : the crop will be destroyed; f : there is a flood.
Let us rewrite the given statement as
'If there is a flood, then the crop will be destroyed'. So, the symbolic form of the given statement is $f \rightarrow c$.

Example 7: Let p and q denote the statements:
p : You drive over 70 km per hour.
q : You get a speeding ticket.

Write the following statements into symbolic forms.

(i) You will get a speeding ticket if you drive over 70 km per hour.

(ii) Driving over 70 km per hour is sufficient for getting a speeding ticket.

(iii) If you do not drive over 70 km per hour then you will not get a speeding ticket.
(iv) Whenever you get a speeding ticket, you drive over 70 km per hour.
Solution: (i) $p \rightarrow q$ (ii) $p \rightarrow q$ (iii) $p \rightarrow q$ (iv) $q \rightarrow p$.

Notes: 1. In ordinary language, it is customary to assume some kind of relationship between the antecedent and the consequent in using the conditional. But in logic, the antecedent and the consequent in a conditional statement are not required to refer to the same subject matter. For example, the statement 'If I get sufficient money then I shall purchase a high-speed computer' sounds reasonable. On the other hand, a statement such as 'If I purchase a computer then this pen is red' does not make sense in our conventional language. But according to the definition of conditional, this proposition is perfectly acceptable and has a truth-value which depends on the truth-values of the component statements.

2. Some of the alternative terminologies used to express $p \rightarrow q$ (if p, then q) are the following: (*i*) p implies q

(*ii*) p only if q ('If p, then q' formulation emphasizes the antecedent, whereas 'p only if q' formulation emphasizes the consequent. The difference is only stylistic.)

(*iii*) q if p, or q when p.

(*iv*) q follows from p, or q whenever p.

(*v*) p is sufficient for q, or a sufficient condition for q is p. (*vi*) q is necessary for p, or a necessary condition for p is q. (*vii*) q is consequence of p.

Converse, Inverse and Contrapositive

If $P \rightarrow Q$ is a conditional statement, then

(1). $Q \rightarrow P$ is called its *converse*

(2). $\neg P \rightarrow \neg Q$ is called its *inverse*

(3). $\neg Q \rightarrow \neg P$ is called its *contrapositive*.

Truth table for $Q \rightarrow P$ (converse of $P \rightarrow Q$)

P	Q	$Q \rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for $\neg P \rightarrow \neg Q$ (inverse of $P \rightarrow Q$)

P	Q	$\neg P$	$\neg Q$	$\neg P \rightarrow \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Truth table for $\neg Q \rightarrow \neg P$ (contrapositive of $P \rightarrow Q$)

P	Q	$\neg Q$	$\neg P$	$\neg Q \rightarrow \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Example: Consider the statement

P : It rains.

Q: The crop will grow.

The implication $P \to Q$ states that

R: If it rains then the crop will grow.

The converse of the implication $P \to Q$, namely $Q \to P$ sates that S: If the crop will grow then there has been rain.

The inverse of the implication $P \to Q$, namely $\neg P \to \neg Q$ sates that

U: If it does not rain then the crop will not grow.

The contraposition of the implication $P \to Q$, namely $\neg Q \to \neg P$ states that T : If the crop do not grow then there has been no rain.

Example 9: *Construct the truth table for* $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

Biconditional proposition

If p and q are any two statements (propositions), then the statement $p \leftrightarrow q$ which is read as 'p if and only if q' and abbreviated as 'p iff q' is called a **biconditional statement** and the connective is the **biconditional connective.**

The truth table of p↔q is given by the following table:

Table 6. Truth table for biconditional

p	q	p↔q
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*T*

It may be noted that p q is true only when both p and q are true or when both p and q are false. Observe that p q is true when both the conditionals $p \to q$ and $q \to p$ are true, *i.e.*, the truth-values of $(p \to q) \wedge (q \to p)$, given in Ex. 9, are identical to the truth-values of p q defined here.

Note: The notation $p \leftrightarrow q$ is also used instead of p↔q.

TAUTOLOGY AND CONTRADICTION

Tautology: A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a **universally valid formula** or a **logical truth** or a **tautology.**

Contradiction: A statement formula which is false regardless of the truth values of the statements which replace the variables in it is said to be a **contradiction.**

Contingency: A statement formula which is neither a tautology nor a contradiction is known as a **contingency.**

Substitution Instance

A formula A is called a substitution instance of another formula B if A can be obtained form B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

Example: Let $B : P \rightarrow (J \wedge P)$.

Substitute $R \leftrightarrow S$ for P in B, we get

$$(i): (R \leftrightarrow S) \rightarrow (J \wedge (R \leftrightarrow S))$$

Then A is a substitution instance of B.

Note that $(R \leftrightarrow S) \rightarrow (J \wedge P)$ is not a substitution instance of B because the variables

P in $J \wedge P$ was not replaced by $R \leftrightarrow S$.

Equivalence of Formulas

Two formulas A and B are said to equivalent to each other if and only if $A \leftrightarrow B$ is a tautology.

If $A \leftrightarrow B$ is a tautology, we write $A \Leftrightarrow B$ which is read as A is equivalent to B.

Note : 1. $\Leftrightarrow$ is only symbol, but not connective.

2. $A \leftrightarrow B$ is a tautology if and only if truth tables of A and B are the same.
3. Equivalence relation is symmetric and transitive.

Method I. Truth Table Method: One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

Example: Prove $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $P \vee Q$ $\quad$ $\neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Example: Prove $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$	$(P \rightarrow Q)$ $(\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \rightarrow Q)$ $\quad$ $(\neg P \vee Q)$ is a tautology then $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Equivalence Formulas:

1. Idempotent laws:

(a) $P \vee P \Leftrightarrow P$ (b) $P \wedge P \Leftrightarrow P$

2. Associative laws:

(a) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (b) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative laws:

(a) $P \vee Q \Leftrightarrow Q \vee P$ (b) $P \wedge Q \Leftrightarrow Q \wedge P$

4. Distributive laws:

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. Identity laws:

(a) (i) $P \vee F \Leftrightarrow P$ (ii) $P \vee T \Leftrightarrow T$

(b) (i) $P \wedge T \Leftrightarrow P$ (ii) $P \wedge F \Leftrightarrow F$

6. Component laws:

(a) (i) $P \vee \neg P \Leftrightarrow T$ (ii) $P \wedge \neg P \Leftrightarrow F$.

(b) (i) $\neg\neg P \Leftrightarrow P$ (ii) $\neg T \Leftrightarrow F$, $\neg F \Leftrightarrow T$

7. Absorption laws:

(a) $P \vee (P \wedge Q) \Leftrightarrow P$ (b) $P \wedge (P \vee Q) \Leftrightarrow P$

8. Demorgan's laws:

(a) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ (b) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Method II. Replacement Process: Consider a formula $A : P \rightarrow (Q \rightarrow R)$. The formula $Q \rightarrow R$ is a part of the formula A. If we replace $Q \rightarrow R$ by an equivalent formula $\neg Q \vee R$ in A, we get another formula $B : P \rightarrow (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A as the replacement process.

Example: Prove that $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$.(May. 2010)

Solution: $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R)$ $[\because Q \rightarrow R \Leftrightarrow \neg Q \vee R]$

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ [by Associative laws]

$\Leftrightarrow \neg(P \wedge Q) \vee R$ [by De Morgan's laws]

$\Leftrightarrow (P \wedge Q) \rightarrow R[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$.

Example: Prove that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$.

Solution: $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$

$\Leftrightarrow (\neg P \wedge \neg R) \vee Q \Leftrightarrow$

$\neg(P \vee R) \vee Q \Leftrightarrow P \vee$

$R \rightarrow Q$

Example: Prove that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

Solution: $P \to (Q \to P) \Leftrightarrow \neg P \vee (Q \to P)$

$\Leftrightarrow \neg P \vee (\neg Q \vee P)$

$\Leftrightarrow (\neg P \vee P) \vee \neg Q$

$\Leftrightarrow T \vee \neg Q$

$\Leftrightarrow T$

and

$\neg P \to (P \to Q) \Leftrightarrow \neg(\neg P) \vee (P \to Q)$

$\Leftrightarrow P \vee (\neg P \vee Q) \Leftrightarrow$

$(P \vee \neg P) \vee Q \Leftrightarrow T$

$\vee Q$

$\Leftrightarrow T$

So, $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

***Example: Prove that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$. (Nov. 2009)

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [Associative and Distributive laws]

$\Leftrightarrow (\neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [De Morgan's laws]

$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$ [Distributive laws]

$\Leftrightarrow T \wedge R$ [$\because \neg P \vee P \Leftrightarrow T$]

$\Leftrightarrow R$

**Example: Show $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is tautology.

Solution: By De Morgan's laws, we have

$$\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$$

$$\neg P \vee \neg R \Leftrightarrow \neg(P \wedge R)$$

Therefore

$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \wedge R)$

$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$

Also

$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$

$\Leftrightarrow P \vee (Q \wedge R)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Hence $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \Leftrightarrow (P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Thus $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$

$\Leftrightarrow T$

Hence the given formula is a tautology.

Example: Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology. (Nov. 2009)

Solution: $(P \wedge Q) \rightarrow (P \vee Q) \Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q)$ [$\because P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ [by De Morgan's laws]

$\Leftrightarrow (\neg P \vee P) \vee (\neg Q \vee Q)$ [by Associative laws and commutative laws]

$\Leftrightarrow (T \vee T)$[by negation laws]

$\Leftrightarrow T$

Hence, the result.

Example: Write the negation of the following statements.

(a). Jan will take a job in industry or go to graduate school.
(b). James will bicycle or run tomorrow.
(c). If the processor is fast then the printer is slow.

Solution: (a). Let P : Jan will take a job in industry.
Q: Jan will go to graduate school.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: Jan will not take a job in industry and he will not go to graduate school.

(b). Let P : James will bicycle.
Q: James will run tomorrow.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: James will not bicycle and he will not run tomorrow.

(c). Let P : The processor is fast.
Q: The printer is slow.

The given statement can be written in the symbolic as $P \rightarrow Q$.

The negation of $P \rightarrow Q$ is given by $\neg(P \rightarrow Q)$.

$$\neg(P \rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow P \wedge \neg Q.$$

$P \wedge \neg Q$: The processor is fast and the printer is fast.

Example: Use Demorgans laws to write the negation of each statement.

(a). I want a car and worth a cycle.
(b). My cat stays outside or it makes a mess.
(c). I've fallen and I can't get up.
(d). You study or you don't get a good grade.

Solution: (a). I don't want a car or not worth a cycle.
(b). My cat not stays outside and it does not make a mess.

(c). I have not fallen or I can get up.

(d). You can not study and you get a good grade.

Exercises: 1. Write the negation of the following statements.

(a). If it is raining, then the game is canceled.

(b). If he studies then he will pass the examination.

Are $(p \to q) \to r$ and $p \to (q \to r)$ logically equivalent? Justify your answer by using the rules of logic to simply both expressions and also by using truth tables. Solution: $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent because

Method I: Consider

$$(p \to q) \to r \Leftrightarrow (\neg p \vee q) \to r$$

$$\Leftrightarrow \neg(\neg p \vee q) \vee r \Leftrightarrow$$

$$(p \wedge \neg q) \vee r$$

$$\Leftrightarrow (p \wedge r) \vee (\neg q \wedge r)$$

and

$$p \to (q \to r) \Leftrightarrow p \to (\neg q \vee r)$$

$$\Leftrightarrow \neg p \vee (\neg q \vee r) \Leftrightarrow$$

$$\neg p \vee \neg q \vee r.$$

Method II: (Truth Table Method)

p	q	r	$p \to q$	$(p \to q) \to r$		$q \to r$	$p \to (q \to r)$
T	T	T	T		T	T	T
T	T	F	T		F	F	F
T	F	T	F		T	T	T
T	F	F	F		T	T	T
F	T	T	T		T	T	T
F	T	F	T		F	F	T
F	F	T	T		T	T	T
F	F	F	T		F	T	T

Here the truth values (columns) of $(p \to q) \to r$ and $p \to (q \to r)$ are not identical.

Consider the statement: "If you study hard, then you will excel". Write its converse, contra positive and logical negation in logic.

Duality Law

Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\vee$ and $\wedge$ are called *duals* of each other. If the formula A contains the special variable T or F, then A^*, its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: Write the dual of the following formulas:

(i). $(P \vee Q) \wedge R$ (ii). $(P \wedge Q) \vee T$ (iii). $(P \wedge Q) \vee (P \vee \neg(Q \wedge \neg S))$

Solution: The duals of the formulas may be written as

(i). $(P \wedge Q) \vee R$ (ii). $(P \vee Q) \wedge F$ (iii). $(P \vee Q) \wedge (P \wedge \neg(Q \vee \neg S))$

Result 1: The negation of the formula is equivalent to its dual in which every variable is replaced by its negation.

We can prove

$$\neg A(P_1, P_2, ..., P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, ..., \neg P_n)$$

Example: Prove that (a). $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

(b). $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Solution: (a).$\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$$\begin{aligned}
&\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q) \\
&\Leftrightarrow (P \wedge Q) \vee \neg P \vee Q \\
&\Leftrightarrow ((P \wedge Q) \vee \neg P)) \vee Q \\
&\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q \\
&\Leftrightarrow (T \wedge (Q \vee \neg P)) \vee Q \\
&\Leftrightarrow (Q \vee \neg P) \vee Q \\
&\Leftrightarrow Q \vee \neg P \\
&\Leftrightarrow \neg P \vee Q
\end{aligned}$$

(b). From (a)

$$(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Leftrightarrow \neg P \vee Q$$

Writing the dual

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$$

Tautological Implications

A statement formula A is said to *tautologically imply* a statement B if and only if $A \rightarrow B$ is a tautology.

In this case we write $A \Rightarrow B$, which is read as 'A implies B'.

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

$A \Rightarrow B$ states that $A \rightarrow B$ is tautology.

Clearly $A \Rightarrow B$ guarantees that B has a truth value T whenever A has the truth value T.

One can determine whether $A \Rightarrow B$ by constructing the truth tables of A and B in the same manner as was done in the determination of $A \Leftrightarrow B$. Example: Prove that $(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

Solution:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$(P \to Q) \to (\neg Q \to \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true, $(P \to Q) \to (\neg Q \to \neg P)$ is a tautology.

Hence $(P \to Q) \Rightarrow (\neg Q \to \neg P)$.

In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding conditional leads to the truth value T for the consequent. This procedure guarantees that the conditional becomes tautology, thereby proving the implication.

Example: Prove that $\neg Q \wedge (P \to Q) \Rightarrow \neg P$.

Solution: Assume that the antecedent $\neg Q \wedge (P \to Q)$ has the truth value T, then both $\neg Q$ and $P \to Q$ have the truth value T, which means that Q has the truth value F, $P \to Q$ has the truth value T. Hence P must have the truth value F.

Therefore the consequent $\neg P$ must have the truth value T.

$$\neg Q \wedge (P \to Q) \Rightarrow \neg P.$$

Another method to show $A \Rightarrow B$ is to assume that the consequent B has the truth value F and then show that this assumption leads to A having the truth value F. Then $A \to B$ must have the truth value T.

Example: Show that $\neg(P \to Q) \Rightarrow P$.

Solution: Assume that P has the truth value F. When P has F, $P \to Q$ has T, then $\neg(P \to Q)$ has F. Hence $\neg(P \to Q) \to P$ has T.

$$\neg(P \to Q) \Rightarrow P$$

Other Connectives

We introduce the connectives NAND, NOR which have useful applications in the design of computers.

NAND: The word NAND is a combination of 'NOT' and 'AND' where 'NOT' stands for negation and 'AND' for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then

$$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$$

The connective $\uparrow$ has the following equivalence:

$$P \uparrow P \Leftrightarrow \neg(P \wedge P) \Leftrightarrow \neg P \vee \neg P \Leftrightarrow \neg P.$$

$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg(P \uparrow Q) \Leftrightarrow \neg(\neg(P \wedge Q)) \Leftrightarrow P \wedge Q.$

$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow \neg P \uparrow \neg Q \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow P \vee Q.$

NAND is Commutative: Let P and Q be any two statement formulas.

$$(P \uparrow Q) \Leftrightarrow \neg(P \wedge Q)$$
$$\Leftrightarrow \neg(Q \wedge P) \Leftrightarrow$$
$$(Q \uparrow P)$$

$\therefore$ NAND is commutative.

NAND is not Associative: Let P, Q and R be any three statement formulas.

Consider $\uparrow (Q \uparrow R) \Leftrightarrow \neg(P \wedge (Q \uparrow R)) \Leftrightarrow \neg(P \wedge (\neg(Q \wedge R)))$

$$\Leftrightarrow \neg P \vee (Q \wedge R))$$
$$(P \uparrow Q) \uparrow R \Leftrightarrow \neg(P \wedge Q) \uparrow R$$
$$\Leftrightarrow \neg(\neg(P \wedge Q) \wedge R) \Leftrightarrow$$
$$(P \wedge Q) \vee \neg R$$

Therefore the connective $\uparrow$ is not associative.

NOR: The word NOR is a combination of 'NOT' and 'OR' where 'NOT' stands for negation and 'OR' for the disjunction. It is denoted by the symbol $\downarrow$.

If P and Q are two formulas then

$$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$$

The connective $\downarrow$ has the following equivalence:

$P \downarrow P \Leftrightarrow \neg(P \vee P) \Leftrightarrow \neg P \wedge \neg P \Leftrightarrow \neg P.$

$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg(P \downarrow Q) \Leftrightarrow \neg(\neg(P \vee Q)) \Leftrightarrow P \vee Q.$

$(P \downarrow P) \downarrow (Q \downarrow Q) \Leftrightarrow \neg P \downarrow \neg Q \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow P \wedge Q.$

NOR is Commutative: Let P and Q be any two statement formulas.

$$(P \downarrow Q) \Leftrightarrow \neg(P \vee Q)$$
$$\Leftrightarrow \neg(Q \vee P) \Leftrightarrow$$
$$(Q \downarrow P)$$

$\therefore$ NOR is commutative.

NOR is not Associative: Let P, Q and R be any three statement formulas. Consider

$$P \downarrow (Q \downarrow R) \Leftrightarrow \neg(P \vee (Q \downarrow R))$$
$$\Leftrightarrow \neg(P \vee (\neg(Q \vee R)))$$
$$\Leftrightarrow \neg P \wedge (Q \vee R)$$
$$(P \downarrow Q) \downarrow R \Leftrightarrow \neg(P \vee Q) \downarrow R$$
$$\Leftrightarrow \neg(\neg(P \vee Q) \vee R) \Leftrightarrow$$
$$(P \vee Q) \wedge \neg R$$

Therefore the connective $\downarrow$ is not associative.

Evidently, $P \uparrow Q$ and $P \downarrow Q$ are duals of each other.

Since

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

Example: Express $P \downarrow Q$ interms of $\uparrow$ only.
Solution:

$$\begin{aligned} \downarrow Q &\Leftrightarrow \neg(P \vee Q) \\ &\Leftrightarrow (P \vee Q) \uparrow (P \vee Q) \\ &\Leftrightarrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \uparrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \end{aligned}$$

Example: Express $P \uparrow Q$ interms of $\downarrow$ only. (May-2012)
Solution:

$$\begin{aligned} \uparrow Q &\Leftrightarrow \neg(P \wedge Q) \\ &\Leftrightarrow (P \wedge Q) \downarrow (P \wedge Q) \\ &\Leftrightarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \downarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \end{aligned}$$

Truth Tables

Example: Show that $(A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$. (May-2012)
Solution: We prove this by constructing truth table.

A	B	$A \oplus B$	$A \downarrow B$	$(A \oplus B) \vee (A \downarrow B)$	$A \uparrow B$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	T	T

As columns $(A \oplus B) \vee (A \downarrow B)$ and $(A \uparrow B)$ are identical.

$\therefore (A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$.

Normal Forms

If a given statement formula $A(p_1, p_2, ...p_n)$ involves n atomic variables, we have 2^n possible combinations of truth values of statements replacing the variables.

The formula A is a tautology if A has the truth value T for all possible assignments of the truth values to the variables $p_1, p_2, ...p_n$ and A is called a contradiction if A has the truth value F for all possible assignments of the truth values of the n variables. A is said to be *satis able* if A has the truth value T for atleast one combination of truth values assigned to $p_1, p_2, ...p_n$.

The problem of determining whether a given statement formula is a Tautology, or a Contradiction is called a decision problem.

The construction of truth table involves a finite number of steps, but the construc-tion may not be practical. We therefore reduce the given statement formula to normal form and find whether a given statement formula is a Tautology or Contradiction or atleast satisfiable.

It will be convenient to use the word ”product” in place of ”conjunction” and ”sum” in place of ”disjunction” in our current discussion.

A product of the variables and their negations in a formula is called an *elementary product*. Similarly, a sum of the variables and their negations in a formula is called an *elementary sum*.

Let P and Q be any atomic variables. Then P , $\neg P \wedge Q$, $\neg Q \wedge P \quad \neg P$, $P \quad \neg P$, and $Q \wedge \neg P$ are some examples of elementary products. On the other hand, P , $\neg P \vee Q$, $\neg Q \vee P \vee \neg P$, $P \vee \neg P$, and $Q \vee \neg P$ are some examples of elementary sums.

Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product. Thus $\neg Q, \wedge \neg P$, and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Disjunctive Normal Form (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a *disjunctive normal form* of the given formula.

Example: Obtain disjunctive normal forms of

(a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: (a) We have

$$P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$$
$$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

(b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$ [using $R \leftrightarrow S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$)

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

which is the required disjunctive normal form.

Note: The DNF of a given formula is not unique.

Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a *conjunctive normal form* of the given formula.

The method for obtaining conjunctive normal form of a given formula is similar to the one given for disjunctive normal form. Again, the conjunctive normal form is not unique.

Example: Obtain conjunctive normal forms of

$$(a)\ P \wedge (P \to Q); \qquad (b)\ \neg(P \vee Q) \leftrightarrow (P \wedge Q).$$

Solution: (a). $P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

(b). $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q))$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$$

$$\Leftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Note: A given formula is tautology if every elementary sum in CNF is tautology.

Example: Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution: First we obtain a CNF of the given formula.

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q)$$

$$\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Since each of the elementary sum is a tautology, hence the given formula is tautology.

Principal Disjunctive Normal Form

In this section, we will discuss the concept of principal disjunctive normal form (PDNF).

Minterm: For a given number of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

Let P and Q be the two statement variables. Then there are 2^2 minterms given by $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, and $\neg P \wedge \neg Q$.

Minterms for three variables P , Q and R are $P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$ and $\neg P \wedge \neg Q \wedge \neg R$. From the truth tables of these minterms of P and Q, it is clear that

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

(i). no two minterms are equivalent

(ii). Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

Definition: For a given formula, an equivalent formula consisting of disjunctions of minterms only is called the Principal disjunctive normal form of the formula.

The principle disjunctive normal formula is also called the sum-of-products canonical form.

Methods to obtain PDNF of a given formula

(a). By Truth table:

(i). Construct a truth table of the given formula.

(ii). For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of P and Q.

(iii). The disjunction of these minterms will then be equivalent to the given formula.

Example: Obtain the PDNF of $P \to Q$.
Solution: From the truth table of $P \to Q$

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

The PDNF of $P \to Q$ is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

$$\therefore P \to Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

Example: Obtain the PDNF for $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.
Solution:

P	Q	R	Minterm	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \neg R$	T	F	F	T
T	F	T	$P \wedge \neg Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \neg Q \wedge \neg R$	F	F	F	F
F	T	T	$\neg P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\neg P \wedge Q \wedge \neg R$	F	F	F	F
F	F	T	$\neg P \wedge \neg Q \wedge R$	F	T	F	T
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$	F	F	F	F

The PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R).$$

(b). Without constructing the truth table:

In order to obtain the principal disjunctive normal form of a given formula is constructed as follows:

(1). First replace $\rightarrow$, by their equivalent formula containing only $\wedge$, $\vee$ and $\neg$.

(2). Next, negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.

(3). Any elementarily product which is a contradiction is dropped. Minterms are ob-tained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example: Obtain the principal disjunctive normal form of

$(a)\ \neg P \vee Q$; $(b)\ (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

Solution:

(a) $\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T) \quad [\because A \wedge T \Leftrightarrow A]$

$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)) \; [\because P \vee \neg P \Leftrightarrow T]$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$[\because P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \quad [\because P \vee P \Leftrightarrow P]$$

$(b)\ (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$$

$$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q)(\neg P \wedge R \wedge \neg Q)$$

$$\vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$$

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

$$P \vee (P \wedge Q) \Leftrightarrow P$$

$$P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$$

Solution: We write the principal disjunctive normal form of each formula and com-pare these normal forms.

$(a)\ P \vee (P \wedge Q) \Leftrightarrow (P \wedge T) \vee (P \wedge Q) \quad [\because P \wedge Q \Leftrightarrow P]$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \quad [\because P \vee \neg P \Leftrightarrow T]$$

$\Leftrightarrow ((P \wedge Q) \vee (P \wedge \neg Q)) \vee (P \wedge Q)$ [by distributive laws]

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$ $[\because P \vee P \Leftrightarrow P]$

which is the required PDNF.

Now, $\Leftrightarrow P \wedge T$

$$\Leftrightarrow P \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$$

which is the required PDNF.

Hence, $P \vee (P \wedge Q) \Leftrightarrow P$.

$(b)\ P \vee (\neg P \wedge Q) \Leftrightarrow (P \wedge T) \vee (\neg P \wedge Q)$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Now,

$$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Hence, $\quad P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q.$

Example: Obtain the principal disjunctive normal form of

$$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)). \qquad \text{(Nov. 2011)}$$

Solution: Using $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ and De Morgan's law, we obtain

$$\rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)) \Leftrightarrow \neg P$$
$$\vee ((\neg P \vee Q) \wedge (Q \wedge P))$$

$$\Leftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)) \Leftrightarrow$$
$$\neg P \vee F \vee (P \wedge Q)$$

$$\Leftrightarrow \neg P \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge T) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

Hence $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the required PDNF.

Principal Conjunctive Normal Form

The dual of a minterm is called a Maxterm. For a given number of variables, the *maxterm* consists of disjunctions in which each variable or its negation, but not both, appears only once. Each of the maxterm has the truth value F for exactly one com-bination of the truth values of the variables. Now we define the principal conjunctive normal form.

For a given formula, an equivalent formula consisting of conjunctions of the max-terms only is known as its *principle conjunctive normal form*. This normal form is also called the *product-of-sums canonical form.*The method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

Example: Obtain the principal conjunctive normal form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$
Solution:

$$(\neg P \to R) \wedge (Q \leftrightarrow P)$$

$$\Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \to P) \wedge (P \to Q)]$$

$$\Leftrightarrow (P \vee R) \wedge [(\neg Q \vee P) \wedge (\neg P \vee Q)]$$

$$\Leftrightarrow (P \vee R \vee F) \wedge [(\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)]$$

$$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$$

$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$$
$$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

$$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

which is required principal conjunctive normal form.

Note: If the principal disjunctive (conjunctive) normal form of a given formula A containing n variables is known, then the principal disjunctive (conjunctive) normal form of $\neg A$ will consist of the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the principal disjunctive (conjunctive) normal form of A. From $A \Leftrightarrow \neg\neg A$ one can obtain the principal conjunctive (disjunctive) normal form of A by repeated applications of De Morgan's laws to the principal disjunctive (conjunctive) normal form of $\neg A$.

Example: Find the PDNF form PCNF of $S : P \vee (\neg P \to (Q \vee (\neg Q \to R)))$.
Solution:

$$\Leftrightarrow P \vee (\neg P \to (Q \vee (\neg Q \to R)))$$

$$\Leftrightarrow P \vee (\neg(\neg P) \vee (Q \vee (\neg(\neg Q) \vee R))$$

$$\Leftrightarrow P \vee (P \vee Q \vee (Q \vee R)))$$

$$\Leftrightarrow P \vee (P \vee Q \vee R)$$

$$\Leftrightarrow P \vee Q \vee R$$

which is the PCNF.

Now PCNF of $\neg S$ is the conjunction of remaining maxterms, so

PCNF of $\neg S : (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

Hence the PDNF of S is

$\neg$(PCNF of $\neg S$) : $(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$
$\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$